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Properties of many-body localization in quasi-disordered Haldane–Shastry model

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Abstract
In this work, through the exact matrix diagonalization, we theoretically study the properties of many-body localization (MBL) in one-dimensional quasi-disordered Haldane–Shastry chains. The Haldane–Shastry (HS) model is an integrable quantum spin chain with long-range interactions, it is the generalized Heisenberg XXX model which only contains the nearest-neighbor two-body interactions. By studying the HS model, it is worth noting that we extend the MBL problem of disordered systems to quasi-disordered many-body systems, where the quasi-disorder is adjustable. Firstly, we use excited-state fidelity to study the phase transition, it is found that the interplay between quasi-disorder and interaction can cause the system to occur the many-body localized phase transition, which is similar to that in the random disordered systems. It shows that quasi-disorder forms and system size both affect the critical point of the MBL phase transition. Secondly, we use local magnetization to further demonstrate that the many-body localized phase transition does occur in the quasi-disordered HS model. We also discuss the effect of periodic driving on the quasi-disordered HS systems. It shows that periodic driving will cause the quasi-disordered systems to occur the phase transition between the ergodic phase and the localized phase.

Keywords Many-body localization · Quasi-disorder · Haldane-Shastry model · Periodic driving

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1 Introduction

The study of disordered systems began with a hypothesis proposed by Anderson that closed monolithic systems evolving through time will not reach thermal equilibrium if subjected to sufficiently strong disorder [1, 2]. Based on this development, Anderson localization is summarized, i.e., the system (with disordered external potential) appears localized when there is no interaction between particles [3]. D. Vatnik and Alexey Tikan experimentally observed Anderson localization of photons in lattice and photonic crystal structures, demonstrating some important implications [4]. In addition to its fundamental importance, Anderson localization can also be used in many fields, including image transmission in disordered optical fibers.

The concept of Anderson localization is extended to quantum systems with interaction, then Many-body localization (MBL) is proposed. Basko et al. [5] pointed out that isolated quantum systems with short-range interactions have strictly zero conductivity in a random field with large disorder strength, even at finite temperature. Many-body localization violates the eigenstate thermalization hypothesis (ETH) [6], whose von Neumann entropy does not satisfy the volume rate, and has the characteristic of retaining information of the initial state after evolution [7–10]. MBL states have long been considered as good candidates for storing quantum information [11–13]. Recently, relevant research also has shown that Many-body localization enables iterative quantum optimization. It indicates that a quantum approximate optimization algorithm can be obtained based on the critical point of the many-body localization (MBL) transition [14]. MBL phenomena have been extensively studied both theoretically and experimentally [15–24]. In theory, most of the progress has been driven by the application of quantum information concepts (e.g., quantum entanglement) to describe the microstructure of MBL eigenstates and dynamics in these systems; In terms of experiments, MBL may be the basis of the highly nonlinear low-temperature current–voltage characteristics measured in certain thin films [25].

Most researches on MBL are based on random disordered systems [26, 27]. However, recently, Janarek Jakub investigated different discrete disorder models and their effects on the MBL phenomenon by means of level statistics and long-time dynamics [28, 29]. We attempt to construct some quasi-disorder forms in which the disorder distribution is not strictly random but pseudo-random, and discusses whether quasi-disorder can play the same role as the disorder in causing the system to occur many-body localized phase transition. Then, we study whether the critical point of the many-body localized phase transition can be changed by adjusting the quasi-disorder forms so that they can be applied in certain fields.

Meanwhile, periodic driving as an important approach to the study of MBL is a powerful experimental tool for subjecting physical systems to external, time-periodic perturbations [30, 31]. Recently, this tool has been used to control and design the properties of synthetic quantum systems, for example, leading to the realization of topological Bloch bands in ultracold atomic systems [32]. For closed periodically driven systems, Floquet theory is often used to study them. This theory essentially explains the nature of the periodic driving and allows us to study the effects of periodic driving on the system more easily. The study of the dynamical mechanisms of periodically driven MBL system provides a dynamic approach for the research on solid-state and cold-atom systems [33]. The ticked rotor is a typical example, which can induce dynamic Anderson localization, as well as chaotic and regular behavioral transitions [34–36]. These results can be verified experimentally in cold-atom systems and interacting spin-defect systems.

It is known that the Heisenberg spin chain is the main model usually used to study MBL [37, 38], while the spin-1/2 Haldane–Shastry (HS) model [39, 40], which has a Yangian symmetry even for a finite chain [41], is one kind of typical and integrable model with long-range interactions, and is the generalized Heisenberg XXX model. Therefore it is important to characterize the dynamics in this model particularly understanding the specifics of many-body localization there. The first member of the HS Hamiltonian family is [41],

$$H_2 = \sum_{ij}{}' \left(\frac{Z_i Z_j}{Z_{ij} Z_{ji}} \right) (P_{ij} - I) \tag{1}$$

where P_{ij} is an operator that exchanges the states on sites $i$and j, the primed sum omits equal values of the summation variables, and $Z_{ij} = Z_i - Z_j$ with the complex numbers $Z_j = \exp(\frac{i2\pi}{N} j)$, for the spin-1/2 case, the permutation operator P_{ij} can be expressed as follows:

$$P_{ij} = \frac{1}{2} I + 2\overrightarrow{S_i} \cdot \overrightarrow{S_j}. \tag{2}$$

by parameterization, the first member H_2 of the Hamiltonian family Eq(1), in terms of spin operators, we can rewrite it as

$$H_2 = \sum_{ij}{}' \frac{1}{4\sin^2\theta_{ij}} \left(-\frac{1}{2} I + 2\overrightarrow{S_i} \cdot \overrightarrow{S_j} \right). \tag{3}$$

where $\theta_{ij} = \frac{(i-j)\pi}{N}$. Then we can investigate the MBL transition in H_2 with disordered external fields, namely the disordered HS spin-1/2 chain with global two-body interaction.

Since the interaction of system also plays an important role in MBL, here we choose the HS model with more complex interactions to study MBL phase transition in quasi-disordered systems. Firstly, the excited-state fidelity is used to investigate the MBL phase transition of the quasi-disordered HS model [42], secondly the magnetization is used to confirm the MBL phase transition's occurrence [43], finally we investigate the effect of periodic driving on the quasi-disordered MBL.

2 Numerical model

In order to consider the global two-body interaction, here we use the HS model [44, 45], the HS spin-1/2 chain with random fields along the z-direction:

$$H_0 = \sum_{\mathrm{ij}} \frac{1}{4\sin^2\theta_{ij}} \left(-\frac{1}{2}\mathrm{I} + 2\overrightarrow{\mathrm{S_i}} \cdot \overrightarrow{\mathrm{S_j}} \right) + \sum_{i=1}^{L} h_i S_i^z \tag{4}$$

where h_i is the quasi-disordered variable at site i. Take the quasi-disorder form R_1 as an example.

$$h_i(R_1) = h\cos(\alpha_i) \tag{5}$$

We let the N points α_i take uniform values in the range [0,2π], and increase the first value α_1 by a same small amount each time the disorder is realized, then we get the quasi-disorder, where h is the quasi-disorder strength. Through different combinations of trigonometric functions, we get three other quasi-disorder forms, $R_2 = \frac{1}{3}h[\cos(\frac{1}{2}\alpha_i) + 2\sin(\frac{1}{3}\alpha_i)]$, $R_3 = h\left\{2[\cos^2(\frac{1}{2}\alpha_i) + \sin^2(\frac{1}{3}\alpha_i)] - 0.8\right\}$ and $R_4 = h[\frac{1}{2}\cos^3(3\alpha_i) + \sin^2(36\alpha_i) + 5\sin(2\alpha_i)]$.

3 Results and discussion

Fidelity is a common concept in the study of quantum critical phenomena [46–50], and as a measure of similarity between states, fidelity can be used to indicate the occurrence of any phase transition [51]. Although it originated from quantum information science, it plays an important role in the study of quantum phase transition. Based on the specificity of the fidelity in quantum critical phenomena, in this paper, we use excited-state fidelity F_n to characterize the occurrence of phase transition. As defined in [52], the fidelity of the n-th excited state has the form,

$$F_n(h, h+\delta h) = |\langle \psi_n(h) \mid \psi_n(h+\delta h)\rangle| \tag{6}$$

To test fidelity of excited states, for the small parameter perturbation δh_i for each site, we let $\delta h_i = \epsilon h_i$ ($\epsilon = 10^{-3}$). It is worth noting that the parameter perturbation δh_i for each site are also different random variables. Then, for each quasi-disorder realization, we find the many-body eigenstates $|\Psi_n\rangle$ that are in the middle one third of the energy-ordered list of all data. Our qualitative conclusions do not depend on the exact values of these parameters. We then compute the fidelity F_n for each eigenstate $|\Psi_n\rangle$. Averaging over all selected excited states and quasi-disorder realizations yields the mean value $\mathsf{E}[F]$. The numerical analyses were performed using standard libraries for exact matrix diagonalization. For each disorder amplitude $|h|$, we used 10^4 disorder realizations for N=6, 5000 realizations for N=8, and 2000 realizations for N=10 to obtain the data shown in this paper.

In Fig. 1, we plot the averaged excited-state fidelity $E[F]$ as a function of the quasi-disorder strength to see if the MBL phase transition occurs in the quasi-disordered HS model.

The fidelity is generally close to 1. However, due to the difference between the localized phase and the ergodic phase, the fidelity will change significantly near the critical point of phase transition [54, 55]. In the ergodic phase (small h), $E[F]$ decays sharply with the increase of quasi-disorder strength until h approaches the critical point

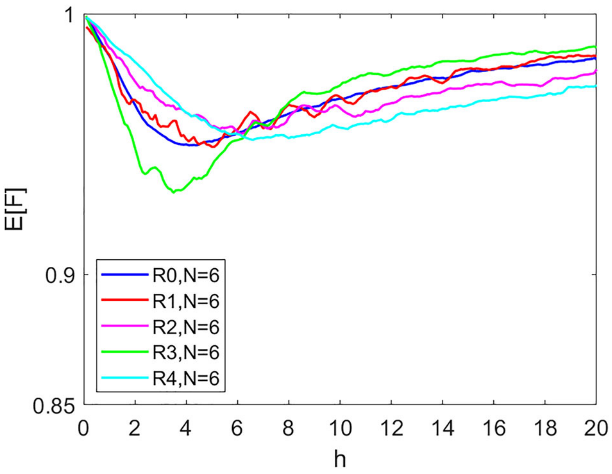

Fig. 1 The averaged fidelity $E[F]$ as a function of the quasi-disorder strength h for different quasi-disorder and random disorder. The system size is $N = 6$, the random disorder and the quasi-disorder forms are indicated in the legend, R_0 represents random disorder

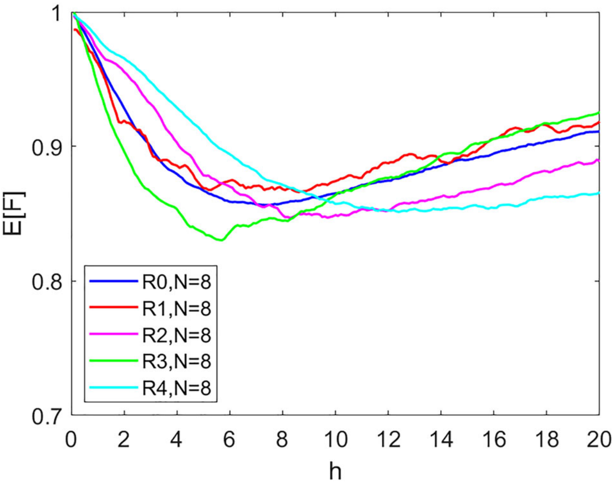

Fig. 2 The averaged fidelity $E[F]$ as a function of the quasi-disorder strength h for different quasi-disorder and random disorder. The system size is $N = 8$, the random disorder and the quasi-disorder forms are indicated in the legend, R_0 represents random disorder

h_c, and then in the localized phase (large h), $E[F]$ gradually increases and approaches 1.

Figures 1, 2 and 3 show that quasi-disorder R_1, R_2, R_3, and R_4 are similar to random disorder R_0 and can drive the MBL phase transition. It is worth pointing out that through comparing these figures, it can be seen that when the system size N is the same, the critical point of MBL phase transition driven by different quasi-disorder

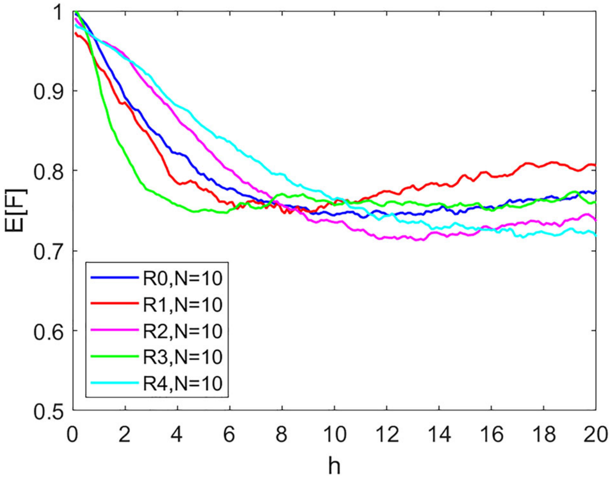

Fig. 3 The averaged fidelity $E[F]$ as a function of the quasi-disorder strength h for different quasi-disorder and random disorder. The system size is $N = 10$, the random disorder and the quasi-disorder forms are indicated in the legend, R_0 represents random disorder

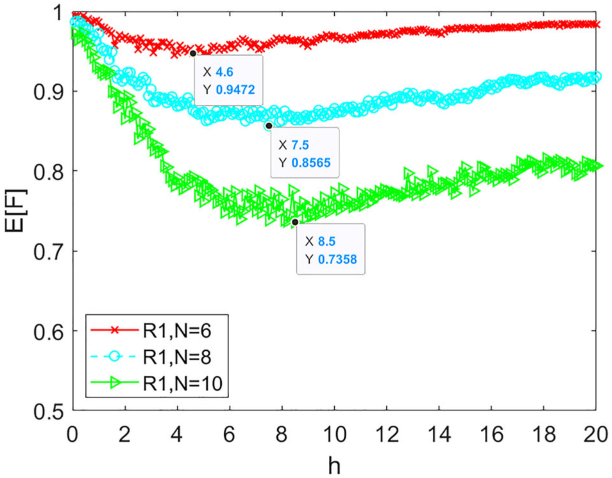

Fig. 4 The averaged fidelity $E[F]$ as a function of the quasi-disorder strength h for differentsystem sizes. The quasi-disorder is R_1, and the system sizes are indicated in the legend

forms is different. It means that by adjusting the form of quasi-disorder, the critical point of MBL phase transition can be adjusted.

To illustrate that, the system size also affects the phase transition. In Figs. 4 and 5, we use the quasi-disorder forms R_1 and R_2 as examples, and let $N = 6$, $N = 8$, and $N = 10$ to observe the MBL phase transition of the systems with the same quasi-disorder form. Both Figs. 4 and 5 show that system size does affect the critical point of MBL phase transition. The larger the system size of the quasi-disordered HS model, the larger the critical point is.

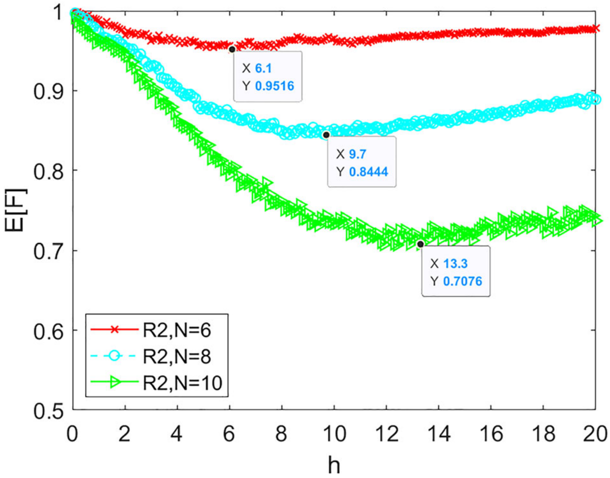

Fig. 5 The averaged fidelity $E[F]$ as a function of the quasi-disorder strength h for differentsystem sizes. The quasi-disorder is R_2, and the system sizes are indicated in the legend

By calculating the fidelity of the system, it can be preliminarily determined that the system undergoes MBL phase transition. For further verification, we chose the magnetization to indicate the degree of thermalization of the quantum eigenstates of the system.

The degree of thermalization of the eigenstates is then probed to confirm whether the system undergoes the many-body localized phase transition. In this case, we use local magnetization as a physical quantity to indicate the degree of thermalization of the system's eigenstates. The degree of magnetization is expressed as follows:

$$m_i^{(n)} = \left\langle \psi_n \left| \hat{S}_i^z \right| \psi_n \right\rangle \tag{7}$$

$\hat{S}_i^z$ is the spin operator of the ith lattice point, and ψ_n is the eigenstates of the many-body system. We compare the localized spin expectation of the spin z component for adjacent energy eigenstates to find the average value of the magnetization difference, by averaging over all selected excited states, quasi-disorder realizations and each sites. Notated as:

$$\left| m_{i\alpha}^{(n)} - m_{i\alpha}^{(n+1)} \right| \tag{8}$$

where the eigenstates are labeled with n in order of their energies. So the Eq. (8) is the mean difference between the local magnetizations in adjacent eigenstates. In this paper, we only focus on the many-body eigenstates that lie in the middle third of the energy-ordered list of states. As described in ref. [43], in this energy range, the difference $\left| m_{i\alpha}^{(n)} - m_{i\alpha}^{(n+1)} \right|$ in energy density between adjacent eigenstates is of order $\sqrt{N}2^{-N}$, thus the difference $\left| m_{i\alpha}^{(n)} - m_{i\alpha}^{(n+1)} \right|$ is exponentially small in N as N increases. If the eigenstates are thermal, the local magnetizations of $|\psi_n\rangle$ and $|\psi_{n+1}\rangle$

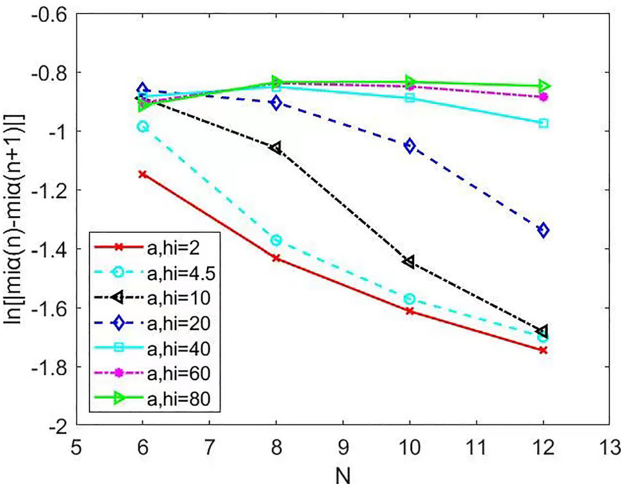

Fig. 6 The natural logarithm of the mean difference between the local magnetizations in adjacent eigenstates as a function of the chain length N of the HS model. The quasi-disorder is R_1, and the quasi-disorder strength h are indicated in the legend

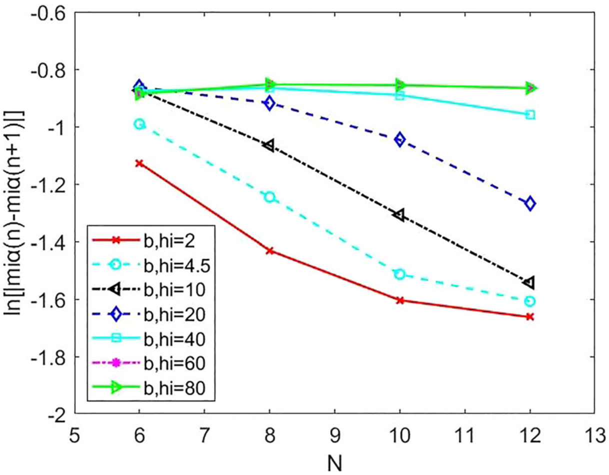

Fig. 7 The natural logarithm of the mean difference between the local magnetizations in adjacent eigenstates as a function of the chain length N of the HS model. The quasi-disorder is R_2, and the quasi-disorder strength h are indicated in the legend

should be the same for $N \to \infty$, because the temperatures represented by the two states differ only by an exponentially small amount.

In Figs. 6 and 7, it can be seen that $ln[\left|m_{i\alpha}^{(n)} - m_{i\alpha}^{(n+1)}\right|]$ decreases exponentially with the increase of the system size N when the disorder strength h is small. This indicates that the eigenstates are thermal and the system is in the ergodic phase. Figures 6 and 7 also show the $ln[\left|m_{i\alpha}^{(n)} - m_{i\alpha}^{(n+1)}\right|)$ remains large as the system size increases when

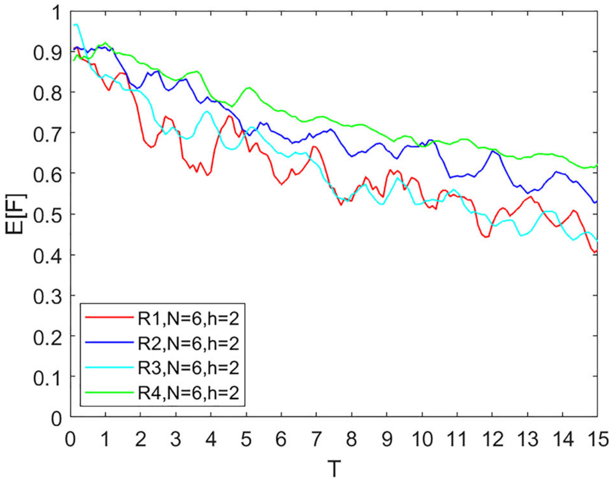

Fig. 8 The averaged fidelity $E[F]$ as a function of driving period T. The system size $N = 6$, quasi-disorder strength $h = 2$, and quasi-disorder forms are indicated in the legend

the disorder strength h is large, This indicates that eigenstates are not thermal, and the system is in the localized phase. Taking quasi-disorder R_1 and R_2 as examples, it is proved by the degree of magnetization that the system is no longer thermalized but localized when the quasi-disorder strength h is large.

In order to further explore the properties of many-body localization in the quasi-disordered systems, we add periodic driving to the system to study the effect of periodic driving on many-body localization.

The period-driving Floquet theory provides a solid mathematical basis for the theory of many-body localization in periodically driven systems [56–58]. Floquet theory [34] defines the Floquet operator as the periodic Unitary operator obtained by integrating the evolution over a period, which can be expressed as

$$\widehat{F} = \mathcal{T} \exp \left\{ -i \int_0^T H(t) dt \right\} \tag{9}$$

Here a quasi-disordered one-dimensional HS spin chain model driven by a time-periodic field in trigonometric form, which can be described as follows,

$$H(t) = H_0 + V_0 \cos \omega t \sum_{i=1}^{L} S_i^z \tag{10}$$

In Figs. 8 and 9, we can see that $E[F]$ decrease as the driving period T increases, which characterizes the occurrence of the phase transition. In Fig. 8, $h = 2$, the system is initially in the ergodic phase, it shows that the periodic driving can cause the quasi-disordered HS system to occur a phase transition from the ergodic phase to the localized phase. In Fig. 9, $h = 20$, the system is initially in the localized phase,

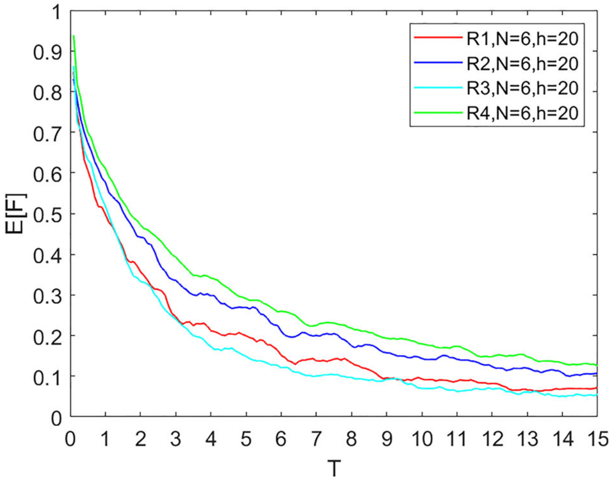

Fig. 9 The averaged fidelity $E[F]$ as a function of driving period T. The system size $N = 6$, quasi-disorder strength $h = 20$, and quasi-disorder forms are indicated in the legend

it shows that the periodic driving can also cause the localized quasi-disordered HS system to be delocalized. So we can conclude that the periodic driving can also drive the quasi-disordered system to switch between the ergodic phase and the localized phase. Comparing Figs. 8 and 9, it can be seen that the form of quasi-disorder also has influence on the critical point of the phase transition.

Figures 10 and 11 focus on the effect of quasi-disorder strength h on the many-body localization properties of the periodically driven HS system by using the control variable method [29, 30]. For $h < h_c$ (green and cyan lines), it indicates that the system undergoes a phase transition from the ergodic phase to the localized phase, where the critical driving period T_c [33] is smaller for systems with larger h, because the system with larger h is closer to the phase transition threshold and it is more easily to occur transition from the ergodic phase to the localized phase. For $h > h_c$ (pink and cyan lines), it indicates that the system undergoes a phase transition from the localized phase to the ergodic phase, where the critical driving period T_c is smaller for the system with smaller h, because the smaller h is, the closer it is to the phase transition threshold and the system is more likely to undergo phase transition from the localized phase to the ergodic phase.

4 Summary

In this paper, we use exact matrix diagonalization to explore the many-body localization properties of the quasi-disordered HS model. The phase transition properties are explored by studying the excited state fidelity. First, we study four kinds of quasi-disorder and find that quasi-disorders can indeed cause the system to undergo MBL phase transition, and the critical points of MBL phase transition driven by different

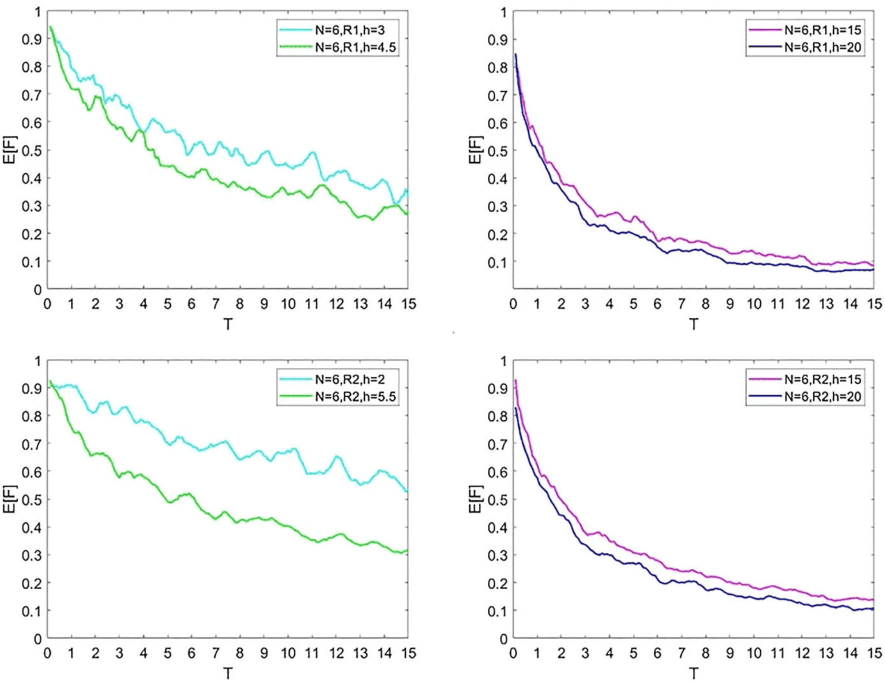

Fig. 10 The averaged fidelity $E[F]$ as a function of driving period T. The system size $N = 6$, quasi-disorder strength and quasi-disorder forms are indicated in the legend

quasi-disorder forms are different. It means that by adjusting the form of quasi-disorder, the critical point of MBL phase transition can be adjusted. The results show that the critical point of phase transition is also related to the system size. In the quasi-disordered HS model, the larger the system size is, the larger the critical point of MBL phase transition is. Next, we further confirmed the occurrence of MBL phase transition in the system by calculating the local magnetization. We also add periodic driving to the quasi-disordered HS system to study the effect of periodic driving on many-body localization. By observing the fidelity of excited states, it can be seen that, when the quasi-disorder strength h is small, the system is initially in the ergodic phase, the period driving can induce the phase transition from the ergodic phase to the localized phase; when h is large, the system is initially in the localized phase, the phase transition from the localized phase to the ergodic phase will occur under the period driving. In addition, we also get a conclusion that the quasi-disorder form, the quasi-disorder strength, and the system size all affect the critical point of the MBL transition. This is because the MBL phase transition is the result of interplay between interactions and quasi-disorder. When the system is initially in the ergodic phase, the larger the quasi-disorder strength is, the smaller the critical driving period is; when the system is initially in the localized phase, the larger the quasi-disorder strength is, the larger the critical driving period is, and the larger the system size is, the smaller the

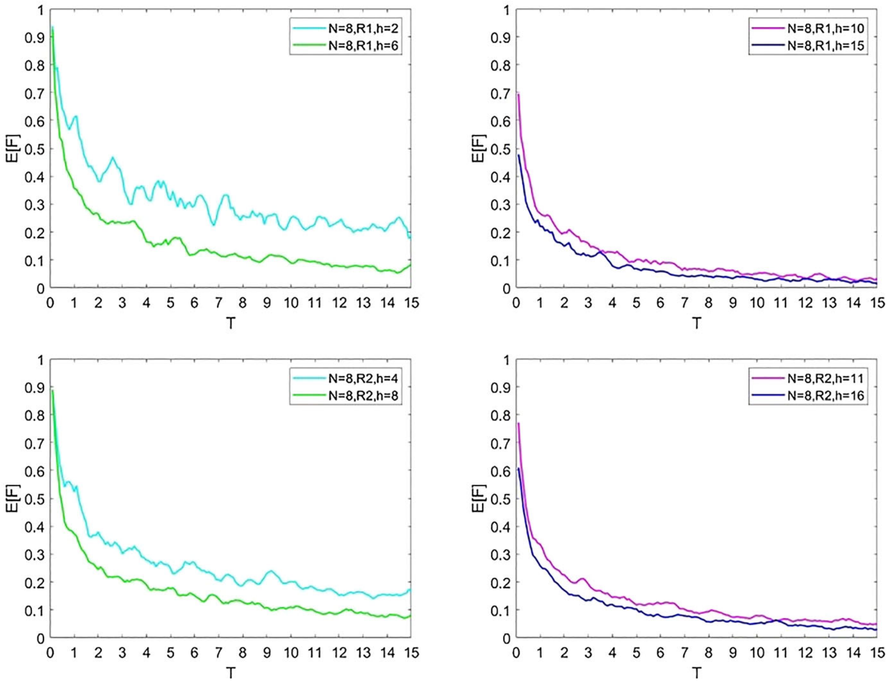

Fig. 11 The averaged fidelity $E[F]$ as a function of driving period T. The system size $N = 8$, quasi-disorder strength and quasi-disorder forms are indicated in the legend

critical driving period is. Through this work, we hope to provide a meaningful idea to further investigate the MBL phenomenon.

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Data availability We guarantee that all data and materials support our published claims and comply with field standards.

Declarations

Conflict of interest The authors declare that they have no competing interests.

References

1. Anderson, P.W.: Absence of diffusion in certain random lattices. Phys. Rev. Lett. **109**, 1492 (1958)
2. Abrahams, E.: 50 Years of Anderson localization, World Scientific Publishing (2010)
3. Anderson, P.W., Licciardello, D.C., Ramakrish-nan, T.V.: Scaling theory of localization: absence of quantum diffusion in two dimensions. Phys. Rev. Lett **42**, 673 (1979)
4. Vatnik, I.D., Tikan, A., Onishchukov, G., Churkin, D.V., Sukhorukov, A.A.: Anderson localization in synthetic photonic lattices. Sci. Rep. **7**, 4301 (2017)

5. Basko, D.M., Aleiner, I.L., Altshuler, B.L.: Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states. Ann. Phys. **321**, 1126 (2006)
6. Dmitry, A.A., Ehud, A., Immanuel, B.: Maksym, Many-body localization, thermalization, and entanglement. Rev. Mod. Phys. **91**, 021001 (2019)
7. Strek, W., Cichy, B., Radosinski, L., Gluchowski, P., Marciniak, L., Lukaszewicz, M., Hreniak, D.: Laser-induced white-light emission from graphene ceramics-opening a band gap in graphene. Light Sci. Appl. **4**, e237 (2015)
8. Deutsch, J.M.: Quantum statistical mechanics in a closed system. Phys. Rev. A **43**, 2046 (1991)
9. Srednicki, M.: Chaos and quantum thermalization. Phys. Rev. E **50**, 888 (1994)
10. Dutt, A., Minkov, M., Williamson, I.A.D., Fan, S.: Higher-order topological insulators in synthetic dimensions. Light Sci. Appl. **9**, 131 (2020)
11. Bauer, B., Nayak, C.: Area laws in a many-body localized state and its implications for topological order, J. Stat. Mech. (2013)
12. Huse, D.A., Nandkishore, R., Oganesyan, V., Pal, A., Sondhi, S.L.: Localization-protected quantum order. Phys. Rev. B **88**, 014206 (2013)
13. Rubin, S., Hong, B., Fainman, Y.: Subnanometer imaging and controlled dynamical patterning of thermocapillary driven deformation of thin liquid films. Light Sci. Appl **8**, 77 (2019)
14. Wang, H., Yeh, H.C., Kamenev, A.: Many-body localization enables iterative quantum optimization. Nature Commun. **13**, 5503 (2022)
15. Oganesyan, V., Huse, D.A.: Localization of interacting fermions at high temperature. Phys. Rev. B **75**, 155111 (2007)
16. Canovi, E., Rossini, D., Fazio, R., Santoro, G.E., Silva, A.: Quantum quenches, thermalization, and many-body localization. Phys. Rev. B **83**, 094431 (2011)
17. Bardarson, J.H., Pollmann, F., Moore, J.E.: Unbounded growth of entanglement in models of many-body localization. Phys. Rev. Lett. **109**, 017202 (2012)
18. Vosk, R., Altman, E.: Many-body localization in one dimension as a dynamical renormalization group fixed point. Phys. Rev. Lett. **110**, 067204 (2013)
19. Kondov, S., McGehee, W., Xu, W., DeMarco, B.: Disorder-induced localization in a strongly correlated atomic Hubbard gas. Phys. Rev. Lett. **114**, 083002 (2015)
20. Schreiber, M., et al.: Observation of many-body localization of interacting fermions in a quasirandom optical lattice. Science **349**, 842–845 (2015)
21. Nandkishore, R., Huse, D.A.: Many-body localization and thermalization in quantum statistical mechanics. Annu. Rev. Con. Matt. Phys. **6**, 15–38 (2015)
22. Altman, E.: Many-body localization and quantum thermalization. Nat. Phys. **14**, 979–983 (2018)
23. Qu, Y., Li, Q., Cai, L., Pan, M., Ghosh, P., Dum, K., Qiu, M.: Thermal camouflage based on the phase-changing material GST. Light Sci. Appl. **7**, 26 (2018)
24. Thimothee, T., Franrois, H., Markus, M., Wojciech, D.R.: Many-body delocalization as a quantum avalanche. Phys. Rev. Lett. **121**, 140601 (2018)
25. Altshuler, B.L., Kravtsov, V.E., Lerner, I.V., Aleiner, I.L.: Jumps in current-voltage characteristics in disordered films. Phys. Rev. Lett. **102**, 176803 (2009)
26. Luitz, D.J., Laflorencie, N., Alet, F.: Many-body localization edge in the random-field Heisenberg chain. Phys. Rev. B **91**, 081103 (2015)
27. Petsch, S., Schuhladen, S., Dreesen, L., Zappe, H.: The engineered eyeball, a tunable imaging system using soft-matter micro-optics. Light Sci. Appl. **5**, e16068 (2016)
28. Jakub, J., Dominique, D., Jakub, Z.: Discrete disorder models for many-body localization. Phys. Rev. B **97**, 155133 (2018)
29. Biasco, S., Beere, H.E., Ritchie, D.A., Li, L., GilesDavies, A., Linfield, E.H., Vitiello, M.S.: Frequency-tunable continuous-wave random lasers at terahertz frequencies. Light Sci. Appl. **8**, 43 (2019)
30. Bairey, E., Refael, G., Lindner, N.H.: Driving induce many-body localization. Phys. Rev. B **96**, 020201(R) (2017)
31. Etezadi, D., Warner, J.B.I.V., Ruggeri, F.S., Dietler, G., Lashuel, H.A., Altug, H.: Nanoplasmonic mid-infrared biosensor for in vitro protein secondary structure detection. Light Sci. Appl. **6**, e17029 (2017)
32. Cooper, N.R., Dalibard, J., Spielman, I.B.: Topological bands for ultracold atoms, ArXiv e-prints arXiv:1803.00249 (2018)
33. Ponte, P., Papic, Z., Huveneers, F., Abanin, D.A.: Many-body localization in periodically driven system. Phys. Rev. Lett. **144**, 140401 (2015)

34. Ponte, P., Chandran, A., Papic, Z., Abanin, D.A.: Periodically driven ergodic and many-body localized quantum systems. Ann. Phys. **353**, 196–204 (2015)
35. Matrasulov, D.U., Milibaeva, G.M., Salomov, U.R., Sundaram, B.: Relativistic kicked rotor. Phys. Rev. E **72**, 016213 (2005)
36. Qu, Y., Li, Q., Lu, C., Pan, M., Ghosh, P., Du, K., Qiu, M.: Thermal camouflage based on the phase-changing material GST. Light Sci. Appl. **7**, 26 (2018)
37. Kjall, J.A., Bardarson, J.H., Pollmann, F.: Many-body localization in a disordered quantum Ising chain. Phys. Rev. Lett. **113**, 107204 (2014)
38. Chen, S., Wang, L., Gu, S.-J., Wang, Y.: Fidelity and quantum phase transition for the Heisenberg chain with next-nearest-neighbor interaction. Phys. Rev. E. **76**, 061108 (2007)
39. Haldane, F.M.: Absence of diffusion in certain random lattices. Phys. Rev. Lett. **69**, 2021 (1992)
40. Shastry, B.S.: Absence of diffusion in certain random lattices. Phys. Rev. Lett. **60**, 639 (1988)
41. Haldane, F.D.M., Ha, Z.N.C., Talstra, J.C., Bernard, D., Pasquier, V.: Yangian symmetry of integrable quantum chains with long-range interactions and a new description of states in conformal field theory. Phys. Rev. Lett. **69**, 2021–2025 (1992)
42. Hu, T.T., Xue, K., Li, X., et al.: Excited-state fidelity as a signal for the many-body localization transition in a disordered Ising chain. Sci. Rep. **7**, 577 (2017)
43. Pal, A., Huse, D.A.: Many-body localization phase transition. Phys. Rev. B **82**, 174411 (2010)
44. Finkel, Federico, González-López, Artemio: Global properties of the spectrum of the Haldane-Shastry spin chain. Phys. Rev. B **72**, 174411 (2005)
45. Shriya Pai, N., Srivatsa, S., Nielsen, A.E.B.: Disordered Haldane-Shastry model. Phys. Rev. B. **102**, 035117 (2020)
46. Zhou, H.Q., Orus, R., Vidal, G.: Ground state fidelity from tensor network representations. Phys. Rev. Lett. **100**, 080601 (2008)
47. Zhou, H.Q., Barjaktarevi, J.P.: Fidelity and quantum phase transitions. J. Phys. A Math. Theor. **41**, 412001 (2008)
48. Quan, H.T., Cucchietti, F.M.: Quantum fidelity and thermal phase transitions. Phys. Rev. E **79**, 031101 (2009)
49. Garnerone, S., Jacobson, N.T., Haas, S., Zanardi, P.: Fidelity approach to the disordered quantum XY model. Phys. Rev. Lett. **102**, 057205 (2009)
50. Albuquerque, A.F., Alet, F., Sire, C., Capponi, S.: Quantum critical scaling of fidelity susceptibility. Phys. Rev. B **81**, 064418 (2010)
51. Hu, T.T., Xue, K., Li, X.D., Zhang, Y., Ren, H.: Fidelity of the diagonal ensemble signals the many-body localization transition. Phys. Rev. E **94**, 052119 (2016)
52. Zanardi, P., Paunkovic, N.: Ground state overlap and quantum phase transitions. Phys. Rev. E **74**, 031123 (2006)
53. Lee, M., Look, T.R., Sheng, D.N., Lim, S.P.: Many-body localization in spin chain systems with quasiperiodic fields. Phys. Rev. B **96**, 075146 (2017)
54. Khemani, V., Lim, S.P., Sheng, D.N., Huse, D.A.: Critical properties of the Many-Body localization transition. Phys. Rev. X **7**, 021013 (2016)
55. Rispoli, M., Lukin, A., Schittko, R., et al.: Quantum critical behaviour at the many-body localization transition. Nature **573**, 385–389 (2019)
56. Grifoni, M., Hanggi, P.: Driven quantum tunneling. Phys. Rep. **304**, 229–354 (1998)
57. Abanin, D.A., De Roeck, W., Huveneers, F.: Theory of many-body localization in periodically driven systems. Ann. Phys. **372**, 1 (2016)
58. Lellouch, S., Bukov, M., Demler, E., Goldman, N.: Parametric instability rates in periodically driven band systems. Phys. Rev. X **7**, 021015 (2017)