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Optimal control of nonlinear aeroelastic system with non-semi-simple eigenvalues at Hopf bifurcation points

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Abstract

This study discusses an efficient method of the Hopf bifurcation control for nonlinear aeroelastic system. The nonlinear aeroelastic system whose linear part has multiple non-semi-simple eigenvalues at critical point gives rise to Hopf bifurcations. The method of the multiple scales and the well-known linear quadratic regulator method are used to deal with the optimal control of the nonlinear system at Hopf bifurcation points. The modal optimal control equation and modal Riccati equation of the nonlinear system are developed to simplify the computations. The conventional Potter's algorithm is extended to solve modal Riccati equation for the modal Riccati matrix of the Hopf bifurcation control. The first-order approximation solutions are developed, which include the gain vectors and inputs. By the way of optimal control, the admissible control input and trajectory of the linear part of the nonlinear aeroelastic system are obtained to minimize the performance measure. Then, we set the appropriate first-order gain vector to adjust the convergence speed of this nonlinear system.

KEYWORDS

Hopf bifurcation, nonlinear aeroelastic system, non-semi-simple eigenvalues of the center subspace, optimal control, the method of multiple scales

1 | INTRODUCTION

In the last years there has been a great interest to analyze control systems displaying complex dynamics. An emerging research field that has become very stimulating is the bifurcation control. The use of feedbacks to stabilize a system with bifurcation has been studies by several authors. Feedback controller can change the nature of a bifurcation. Verduzco and Alvarez[1] presented a new approach to control a Hopf bifurcation in a class nonlinear systems whose linear approximation has two eigenvalues on the imaginary axis, without assuming that the system is controllable. The center manifold theorem is used to project the dynamics on a two-dimensional manifold, and to design a controller that permits us to decide the stability and direction of the emerging periodic solution. A feedback controller is applied to drive the chaotic states of the system to an appropriately defined reference signal in spite of modeling errors. The equilibrium sets, controllability, and stabilizability of dynamic systems near bifurcation have been studied by using a state-feedback strategy.[2,3] Chen[4] has proposed a state-feedback strategy to control vibrations near bifurcation for both continuous time and discrete time systems. Parametric state-feedback control and the harmonic balance are used to develop a method which delays and stabilizes period doubling bifurcations in nonlinear systems[5]. Different bifurcation control methods and the pole placement method for controlling chaotic systems are presented by Nayfeh.[6] The stability, bifurcation, and chaos of autonomous and nonautonomous nonlinear systems are studied by using linear feedback control.[7]

The controlled center dynamics is a reduced order control system whose dimension is the number of controllable modes and whose stabilizability properties determine the stabilizability properties of the full system. This approach was generalized to the general class of nonlinear systems with any number of uncontrollable modes,[8] it was found that by changing the feedback, the stability properties of the control center dynamics will change, and the stability properties of the full order system will change too.

In vibration optimal control, the continuous Riccati equation plays a fundamental and important role. Many methods for solving the Riccati equation have been proposed. The main algorithm includes the matrix transformation,[9,10] the eigenvector method,[10,11] the Schur method[9] and the iterative algorithm,[12] and so on. For the matrix transformation method, the price for replacing the solution of a set of linear equations is to double the order of the set; for the eigenvector method, it needs to solve all the eigensolutions of the matrix with order *2n* which may be impossible if the order n is very large; for the iterative algorithm, it is simple, but its results depend on the selection of the initial value.

Reference 13 presents a new block simultaneous iterative algorithm for solving Riccati equation using its special feature in optimal shape control. Because in this algorithm the three equations are solved simultaneously and some items are common ones, it can both save the computer memory and raise the computing efficiency.

Recently, there have been several studies about the bifurcation control of aeroelastic systems. For example, the numerical bifurcation analysis of static stall of airfoil and dynamic stall under unsteady perturbation are presented.[14] Chen et al[15] considered an aeroelastic system with two freedoms and presented terminal sliding mode control for aeroelastic systems. The optimal control for nonlinear systems has attracted lots of attention. Chen et al[16] discuss a modal optimal control procedure for defective systems with repeated eigenvalues. Al-Hadithi[17] presented a new method for the estimation of Takagi-Sugeno model-based extended Kalman filter and its application to optimal control for nonlinear systems. In the 1990s, pseudospectral (PS) methods were introduced for solving general nonlinear optimal control problems with constrains.[18,19] The feasibility of the PS method and a set of sufficient conditions for the convergence of the approximated optimal are proved on a key assumption that the optimal controller is at least continuous.[20] Kang et al[21] considered the optimal control of feedback linearizable dynamic systems subject to mixed state and control constrains. The convergence of nonlinear optimal control using pseudospectral method for feedback linearizable systems was proved which extends the results to a more general case that includes discontinuous controls.

Although many important results of the Hopf bifurcation control for the nonlinear system have been obtained as mentioned above, in actual engineering problems, such as the linearization of flutter analysis of aeroelasticity, the dynamic analysis of mobility and graspability of general manipulation systems, may have non-semi-simple purely imaginary eigenvalues at a critical point giving rise to multiple Hopf bifurcations.[22,23] So it is necessary to develop the Hopf bifurcation control for this case.

To this end, we investigate the nonlinear system whose linear part has multiple eigenvalues at a critical point, that is λ_i $(i = 1,2,..., m)$ are m multiple eigenvalues, and $Re(\lambda_i) = 0$, $Im(\lambda_i) = \omega_c \neq 0$ $(i = 1,2,..., m)$, respectively. It is well known that if $A_m = G_m$ where A_m is the algebra multiplicity of the eigenvalue λ, and G_m is the number of the linearly independent eigenvectors corresponding to λ, λ is a semi-simple eigenvalues which forms a stable subspace; if $A_m > G_m$, λ is a non-semi-simple eigenvalues which forms a center subspace. In this case the nonlinear system is unstable and Hopf bifurcation occurs as the control parameter passes through a critical value. Such case describes the behaviors of the physical system arising in the flutter analysis of aeroelasticity.

Al-Hadithi[17] shows that in order to control the Hopf bifurcation, any control method through state feedback control can be used. For example, by using the pole assignment method the desired characteristic of the system can be achieved. However, this method can lead to systems where the control action exceeds the allowed environment limits. Selecting closed loop pole with great negative real parts makes the dynamic response of the system to be quick, and the control effort to be greater than permissible levels. The optimal control can be used for optimal selection of closed loop poles.

A few studies can be found to develop the feedback control for the nonlinear aeroelastic system where linear part has multiple non-semi-simple eigenvalues at critical point giving rise to Hopf bifurcations.

In this study, an efficient method for the Hopf bifurcation control of nonlinear aeroelastic system is presented. The method of the multiple scales and the well-known linear quadratic regulator method are used to deal with the optimal control of the nonlinear system. The conventional Potter's algorithm[10] is extended to solve modal Riccati equation for the modal Riccati matrix of the Hopf bifurcation control. The first-order approximation solutions are developed, which include the gain vectors $\mathbf{g}_0$ and $\mathbf{g}_1$, and inputs z_0 and z_1. The present method is based on the Jordan form which is the simple one, the modal optimal control equation and modal Riccati equation of the nonlinear system are developed to simplify the computations.

The contents of the present paper are organized as follows. In Section 2 the nonlinear differential equation for the aeroelastic system and the generalized eigenvalues problem are given. The Section 3 presents the method of the multiple scales in modal optimal control of the nonlinear system. The Section 4 develops the modal optimal control of nonlinear aeroelastic system with non-semi-simple eigenvalues at the critical point of the Hopf bifurcation. An example of an airfoil model is given to show the application and validity of the present methods in the Section 5.

2 | TECHNICAL BACKGROUND

Consider the flutter problem of an airfoil, the nonlinear differential equation of motion[24] is

$$\boldsymbol{M}_0\ddot{\boldsymbol{q}} + (\boldsymbol{K}_0 + \boldsymbol{H}_0)\boldsymbol{q} = \varepsilon Q_0(\boldsymbol{q}), \tag{1}$$

where $\boldsymbol{q}$ is displacement vector, $\boldsymbol{M}_0$ is the mass matrix, $\boldsymbol{K}_0$ the stiffness matrix, $\boldsymbol{H}_0$ the asymmetric aerodynamic matrix, $\boldsymbol{Q}_0$ $(\boldsymbol{q})$ the nonlinear elastic force, ε the small parameter.

The linear part of the nonlinear Equation (1) is

$$\boldsymbol{M}_0\ddot{\boldsymbol{q}} + (\boldsymbol{K}_0 + \boldsymbol{H}_0)\boldsymbol{q} = \boldsymbol{0}. \tag{2}$$

Using the state transformation, the nonlinear Equation (1) and its linear part are as follows

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \varepsilon f(\boldsymbol{x}), \tag{3}$$

and

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \tag{4}$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & -\boldsymbol{M}_0^{-1}(\boldsymbol{K}_0 + \boldsymbol{H}_0) \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}, \tag{5}$$

and x is state vector, $\dot{\boldsymbol{x}}$ is speed vector,

$$\boldsymbol{x} = [\dot{\boldsymbol{q}}^T, \boldsymbol{q}^T]^T. \tag{6}$$

For the Hopf bifurcation analysis, one needs to define the eigenvalue problems of the state matrix $\boldsymbol{A}$ and its adjoint matrix $\boldsymbol{A}^H$ as follows

$$\boldsymbol{AU} = \boldsymbol{UJ}, \boldsymbol{A^H V} = \boldsymbol{VJ^H}, \boldsymbol{V^H U} = I, \tag{7}$$

where matrix $\boldsymbol{J}$ is the Jordan canonical form of $\boldsymbol{A}$, matrix $\boldsymbol{A}^H$ is conjugate transpose of $\boldsymbol{A}$, $\boldsymbol{U}$, and $\boldsymbol{V}$ are the right and left modal matrices of the state matrix $\boldsymbol{A}$.

It is well known that eigenvalues of $\boldsymbol{A}(\boldsymbol{p})$ are the functions of the parameter $\boldsymbol{p}$, and denoted as $\lambda_i(\boldsymbol{p})$ $(i = 1,2,..., m)$. This study discusses the case where $\boldsymbol{A}(\boldsymbol{p}_c)$ has multiple eigenvalues, that is $\lambda_i(\boldsymbol{p}_c)$, are m multiple eigenvalues, and $Re(\lambda_i) = 0$, $Im(\lambda_i) = \omega_c \neq 0$ $(i = 1,2,..., m)$, respectively. Assume that A_m is used to denote the algebra multiplicity of the eigenvalue λ, and G_m is the number of the linearly independent eigenvectors corresponding to λ. If $A_m > G_m$, the λ is a non-semi-simple eigenvalue, then the $m \times m$ Jordan canonical form of $\boldsymbol{A}$ can be written in the form:

$$\boldsymbol{J} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}_{m\times m}, \tag{8}$$

and the system is unstable at the critical point $(\boldsymbol{p}_c, \omega_c)$ of Hopf bifurcation.

3 | MODAL OPTIMAL CONTROL EQUATIONS OF NONLINEAR AIRFOIL WITH NON-SEMI-SIMPLE EIGENVALUES

For the sake of simplicity, this study discusses the case where the state matrix $\boldsymbol{A}$ has two pairs of multiple non-semi-simple eigenvalues at the critical point, that is, the Jordan block $\boldsymbol{J}$ is the canonical form of $\boldsymbol{A}$ can be written in the forms

$$\boldsymbol{J}=\begin{bmatrix}\boldsymbol{J}^{(1)} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{J}^{(2)}\end{bmatrix}_{2m\times 2m},\boldsymbol{J}^{(1)}=\begin{bmatrix}\lambda_1 & 1 & 0 & \cdots & 0\\ 0 & \lambda_1 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & 0 & \lambda_1 & 1\\ 0 & 0 & 0 & 0 & \lambda_1\end{bmatrix}_{m\times m},\boldsymbol{J}^{(2)}=\begin{bmatrix}\lambda_2 & 1 & 0 & \cdots & 0\\ 0 & \lambda_2 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & 0 & \lambda_2 & 1\\ 0 & 0 & 0 & 0 & \lambda_2\end{bmatrix}_{m\times m}. \tag{9}$$

So the system forms two pairs of m-dimensional center subspace, the solution space of the linear approximation can be split into two unstable spaces. Consequently, the nonlinear system has two pairs of m-dimensional center manifold. In this case, the control design should be introduced.

3.1 | The method of the multiple scales in modal optimal control of the nonlinear system at the critical points

The single-input control equation of nonlinear system (3) is given by

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\varepsilon\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{b}z(t), z(t)=\boldsymbol{G}^T\boldsymbol{x}, \tag{10}$$

where the matrix $\boldsymbol{A}$ is the state matrix of the linear approximation, $\boldsymbol{x}\in\boldsymbol{R}^{n\times 1}$ the state vector, $\varepsilon\boldsymbol{f}(\boldsymbol{x})$ the nonlinear elastic force, ε the small parameter, $z(t)$ is the control input, $\boldsymbol{b}\in\boldsymbol{R}^{n\times 1}$ is called the actuator distribution matrix indicating the locations of control forces, $\boldsymbol{G}$ is the modal gain vector.

Introducing the time scales in the method of multiple scales

$$T_r=\varepsilon^r t, r=0,1,2,\ldots,$$

the first approximation of Equation (10) can be expressed as

$$\left.\begin{array}{l}\boldsymbol{x}(t)=\boldsymbol{x}_0(T_0,T_1)+\varepsilon\boldsymbol{x}_1(T_0,T_1)\\ z(t)=z_0(T_0,T_1)+\varepsilon z_1(T_0,T_1)\end{array}\right\}. \tag{11}$$

The derivative in respect of t is transformed to

$$\frac{d}{dt}=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1}=D_0+\varepsilon D_1, \tag{12}$$

since Equation (10) contains a small parameter ε and the gain vector $\boldsymbol{G}$ depends on the small parameter ε, then the gain vector can be expressed in the form of the power series in ε

$$\boldsymbol{G}^T=\boldsymbol{G}_0^T+\varepsilon\boldsymbol{G}_1^T+\varepsilon^2\boldsymbol{G}_2^T+\cdots, \tag{13}$$

and the first approximation of Equation (13) can be expressed as

$$\boldsymbol{G}^T=\boldsymbol{G}_0^T+\varepsilon\boldsymbol{G}_1^T, \tag{14}$$

where $\boldsymbol{G}_0$ is zero-order of the modal gain vector and $\boldsymbol{G}_1$ is the first-order of the modal gain vector.

Using Equations (11), (12), and (14), Equation (10) becomes

$$(D_0+\varepsilon D_1)(\boldsymbol{x}_0+\varepsilon\boldsymbol{x}_1)=\boldsymbol{A}(\boldsymbol{x}_0+\varepsilon\boldsymbol{x}_1)+\varepsilon\boldsymbol{f}(\boldsymbol{x}_0+\varepsilon\boldsymbol{x}_1)+\boldsymbol{b}(\boldsymbol{z}_0+\varepsilon\boldsymbol{z}_1). \tag{15}$$

Equating coefficients of the like powers of ε yields

$$\varepsilon^0 : D_0\boldsymbol{x}_0 = \boldsymbol{A}\boldsymbol{x}_0 + bz_0, \tag{16}$$

$$\varepsilon^1 : D_0\boldsymbol{x}_1 = \boldsymbol{A}\boldsymbol{x}_1 - D_1\boldsymbol{x}_0 + f(\boldsymbol{x}_0) + bz_1, \tag{17}$$

where

$$\begin{aligned} z_0 &= \boldsymbol{G}_0^T\boldsymbol{x}_0, \\ z_1 &= \boldsymbol{G}_0^T\boldsymbol{x}_1 + \boldsymbol{G}_1^T\boldsymbol{x}_0. \end{aligned} \tag{18}$$

From Equation (16), the zero-order approximation solutions z_0 and $\boldsymbol{x}_0$ can be obtained, and the first-order modification solutions z_1 and $\boldsymbol{x}_1$ can be obtained from Equation (17).

3.2 | Modal control equations

In this section it will be assume that $\boldsymbol{A}$ is an $2\,m \times 2\,m$ state matrix, and the nonsingular generalized modal matrix $\boldsymbol{U}$ of $\boldsymbol{A}$ satisfies the equation

$$\boldsymbol{AU} = \boldsymbol{UJ}, \tag{19}$$

where the Jordan block $\boldsymbol{J}$ is expressed by Equation (9), and the generalized modal matrix of the state matrix $\boldsymbol{A}$ is

$$\boldsymbol{U} = [\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}]. \tag{20}$$

Each of the $2\,m \times m$ submatrices $\boldsymbol{U}^{(j)}$ has the form

$$\boldsymbol{U}^{(j)} = [\boldsymbol{u}_1^{(j)}, \boldsymbol{u}_2^{(j)}, \cdots, \boldsymbol{u}_m^{(j)}](j = 1, 2), \tag{21}$$

where $\boldsymbol{u}_i^{(j)}(i = 1, 2, \ldots, m)$ is the ith vector of the submatrix $\boldsymbol{U}^{(j)}$.

The conjugate transpose of $\boldsymbol{A}$ is called adjoined system, that is, for $\boldsymbol{A}^H$, the generalized modes satisfy the following equations

$$\boldsymbol{A}^{\boldsymbol{H}}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{J}^{\boldsymbol{H}}, \boldsymbol{V} = [\boldsymbol{V}^{(1)}, \boldsymbol{V}^{(2)}], \boldsymbol{V}^{(j)} = [\boldsymbol{v}_1^{(j)}, \boldsymbol{v}_2^{(j)}, \cdots, \boldsymbol{v}_m^{(j)}](j = 1, 2), \tag{22}$$

and

$$\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{U} = I. \tag{23}$$

Using the modal transformations

$$\boldsymbol{x}_0 = \boldsymbol{U}\boldsymbol{\xi}_0 = [\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}] \begin{bmatrix} \boldsymbol{\xi}_0^{(1)} \\ \boldsymbol{\xi}_0^{(2)} \end{bmatrix}, \quad \xi_0^{(j)} = [\xi_{01}^{(j)}, \xi_{02}^{(j)}, \ldots, \xi_{0m}^{(j)}]^T,$$

$$\boldsymbol{x}_1 = \boldsymbol{U}\boldsymbol{\xi}_1 = [\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}] \begin{bmatrix} \boldsymbol{\xi}_1^{(1)} \\ \boldsymbol{\xi}_1^{(2)} \end{bmatrix}, \quad \xi_1^{(j)} = [\xi_{11}^{(j)}, \xi_{12}^{(j)}, \ldots, \xi_{1m}^{(j)}]^T (j = 1, 2),$$

where ξ is generalized coordinates vector, the modal control Equations (16) and (17) in the following forms if the direct state feedback control is used,

$$D_0\xi_0 = \begin{bmatrix} \boldsymbol{J}^{(1)} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}^{(2)} \end{bmatrix} \xi_0 + \boldsymbol{b}_1\boldsymbol{g}_0^T\xi_0, \tag{24}$$

$$D_0\xi_1 = \begin{bmatrix} \boldsymbol{J}^{(1)} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}^{(2)} \end{bmatrix} \xi_1 + \boldsymbol{b}_1\boldsymbol{g}_0^T\xi_1 + \boldsymbol{b}_1\boldsymbol{g}_1^T\xi_0 - D_1\xi_0 + \boldsymbol{r}(\xi_0), \tag{25}$$

where

$$
\begin{aligned}
\boldsymbol{b}_1 &= [\boldsymbol{V}^{(1)}, \boldsymbol{V}^{(2)}]^{\boldsymbol{H}}\boldsymbol{b} = [(\boldsymbol{b}_1^{(1)})^T, (\boldsymbol{b}_1^{(2)})^T]^T = [b_{11}^{(1)}, b_{12}^{(1)}, \dots, b_{1m}^{(1)}, b_{11}^{(2)}, b_{12}^{(2)}, \dots, b_{1m}^{(2)}]^T,\\
\boldsymbol{g}_0^T &= \boldsymbol{G}_0^T[\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}] = [(\boldsymbol{g}_0^{(1)})^T, (\boldsymbol{g}_0^{(2)})^T] = [g_{01}^{(1)}, g_{02}^{(1)}, \dots, g_{0m}^{(1)}, g_{01}^{(2)}, g_{02}^{(2)}, \dots, g_{0m}^{(2)}],\\
\boldsymbol{g}_1^T &= \boldsymbol{G}_1^T[\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}] = [g_{11}, g_{12}, \dots, g_{12m}],\\
\boldsymbol{r}(\xi_0) &= \boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{f}(\boldsymbol{U}\xi_0).
\end{aligned} \tag{26}
$$

According to the modal control theory that the constituent Jordan blocks of the matrix $\boldsymbol{J}$ in Equation (24) is controllable by input if and only if $b_{1m}^{(1)} \neq 0$ and $b_{1m}^{(2)} \neq 0$. Since $\boldsymbol{J}^{(1)}$ and $\boldsymbol{J}^{(2)}$ are the Jordan form matrix with order of $(m \times m)$, therefore, the optimal control based on Equations (24) and (25) is simpler than that the original state Equations (16) and (17), and the conventional algorithm for optimal control, such as the Potter's algorithm can be conveniently applied.[10]

4 | MODAL OPTIMAL CONTROL

The optimal control problem can be defined as follow: Determine an admissible control $\mathbf{z}^*(\mathrm{T}_0)$ causing the system Equation (16) to follow an admissible trajectory $\mathbf{x}^*(\mathrm{T}_0)$ in the state space that minimizes the quadratic performance measure

$$
J_0 = \frac{1}{2}\boldsymbol{x}_0^T(T_{0f})\boldsymbol{H}\boldsymbol{x}_0(T_{0f}) + \frac{1}{2}\int_0^{T_{0f}} [\boldsymbol{x}_0^T(T_0)\boldsymbol{Q}\boldsymbol{x}_0(T_0) + \boldsymbol{z}_0^T(T_0)\boldsymbol{R}\boldsymbol{z}_0(T_0)]dT_0, \tag{27}
$$

where $\boldsymbol{H}$ and $\boldsymbol{Q}$ are real symmetric positive semi-definite matrices and $\boldsymbol{R}$ is a real symmetric positive definite matrix. Moreover, we assume that $\boldsymbol{x}_0(T_{0f})$ is free and T_{0f} is fixed. The optimal control problem using the quadratic performance measure can be interpreted as the problem of driving the initial state as close as possible to zero while placing a penalty on the control effort.

Let us consider a system described by the linear state equations (ie, Equation (16))

$$
D_0\boldsymbol{x}_0 = \boldsymbol{A}\boldsymbol{x}_0 + bz_0,
$$

the object is to determine an optimal control minimizing the quadratic performance measure. To this end, we introduce the Hamiltonian

$$
\begin{aligned}
\mathfrak{R}(\boldsymbol{x}_0(T_0), \boldsymbol{z}_0(T_0), \boldsymbol{p}(T_0), T_0) = \frac{1}{2}[\boldsymbol{x}_0^T(T_0)\boldsymbol{Q}\boldsymbol{x}_0(T_0) + \boldsymbol{z}_0^T(T_0)\boldsymbol{R}\boldsymbol{z}_0(T_0)],\\
+ \boldsymbol{p}^T(T_0)[\boldsymbol{A}\boldsymbol{x}_0(T_0) + \boldsymbol{b}\boldsymbol{z}_0(T_0)]
\end{aligned} \tag{28}
$$

where $\boldsymbol{p}(\mathrm{T}_0)$ is an n-vector of lagrange's multipliers know as the co-state vector and whose purpose is to ensure that Equation (16) is taken into account in the minimization process, Then, the necessary conditions for optimality are

$$
\dot{\boldsymbol{x}}_0^*(T_0) = \frac{\partial \mathfrak{R}}{\partial \boldsymbol{p}} = A\boldsymbol{x}_0^*(T_0) + \boldsymbol{b}\boldsymbol{z}_0^*(T_0), \tag{29}
$$

$$
\dot{\boldsymbol{p}}^*(T_0) = -\frac{\partial \mathfrak{R}}{\partial \boldsymbol{x}_0} = -\boldsymbol{Q}\boldsymbol{x}_0^*(T_0) - \boldsymbol{A}^T\boldsymbol{p}^*(T_0), \tag{30}
$$

$$
\boldsymbol{0} = \frac{\partial \mathfrak{R}}{\partial \boldsymbol{z}_0} = \boldsymbol{R}\boldsymbol{z}_0^*(T_0) + \boldsymbol{b}^T\boldsymbol{p}^*(T_0), \tag{31}
$$

$$
\frac{\partial h(T_{0f})}{\partial \boldsymbol{x}_0(T_{0f})} = \boldsymbol{p}^*(T_{0f}), \tag{32}
$$

where $h(\mathrm{T}_{0f})$ is the first term on the right side of Equation (27), that is,

$$h(T_{0f}) = \frac{1}{2}\boldsymbol{x}_0^T(T_{0f})\boldsymbol{H}\boldsymbol{x}_0(T_{0f}). \tag{33}$$

From Equations (32) and (33), yield:

$$\boldsymbol{p}^*(T_{0f}) = \boldsymbol{H}\boldsymbol{x}_0^*(T_{0f}), \tag{34}$$

from Equation (31), we can solve for the optimal control in terms of the co-state to obtain

$$\boldsymbol{z}_0^*(T_0) = -\boldsymbol{R}^{-1}\boldsymbol{b}^T\boldsymbol{p}^*(T_0). \tag{35}$$

It remains to determine the relation between the co-state and the state. We assume that this relation is linear and of the form

$$\boldsymbol{p}^*(T_0) = \boldsymbol{K}(T_0)\boldsymbol{x}_0^*(T_0). \tag{36}$$

Differentiating Equation (36) with respect to time (ie, T_0), we have.

$$\dot{\boldsymbol{p}}^*(T_0) = \dot{\boldsymbol{K}}(T_0)\boldsymbol{x}_0^*(T_0) + \boldsymbol{K}(T_0)\dot{\boldsymbol{x}}_0^*(T_0), \tag{37}$$

so that inserting Equations (29), (30), (35), and (36) into Equation (37), we obtain

$$-\boldsymbol{Q}\boldsymbol{x}_0^*(T_0) - \boldsymbol{A}^T\boldsymbol{K}(T_0)\boldsymbol{x}_0^*(T_0) = \dot{\boldsymbol{K}}(T_0)\boldsymbol{x}_0^*(T_0) + \boldsymbol{K}(T_0)\boldsymbol{A}\boldsymbol{x}_0^*(T_0) - \boldsymbol{K}(T_0)\boldsymbol{b}\boldsymbol{R}^{-1}\boldsymbol{b}^T\boldsymbol{K}(T_0)\boldsymbol{x}_0^*(T_0). \tag{38}$$

Equation (38) can be satisfied for all times provided

$$\dot{\boldsymbol{K}}(T_0) = -\boldsymbol{Q} - \boldsymbol{A}^T\boldsymbol{K}(T_0) - \boldsymbol{K}(T_0)\boldsymbol{A} + \boldsymbol{K}(T_0)\boldsymbol{b}\boldsymbol{R}^{-1}\boldsymbol{b}^T\boldsymbol{K}(T_0), \tag{39}$$

Equation (39) represents a matrix differential equation known as the *Riccati equation*. The solution $\boldsymbol{K}(\mathrm{T}_0)$, called the *Riccati matrix*, is subject to the boundary condition

$$\boldsymbol{K}(T_{0f}) = \boldsymbol{H}(T_{0f}) = \boldsymbol{H}, \tag{40}$$

as can be verified from Equations (34) and (36). If $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{A}$, and $\boldsymbol{b}$ are constant, the modal Riecati matrix $\boldsymbol{K}(T_0) =$ constant, as $\boldsymbol{T}_{0f} \to \infty$. In the case, Equation (39) can be written in the form

$$-\boldsymbol{Q} - \boldsymbol{A}^T\boldsymbol{K}(T_0) - \boldsymbol{K}(T_0)\boldsymbol{A} + \boldsymbol{K}(T_0)\boldsymbol{b}\boldsymbol{R}^{-1}\boldsymbol{b}^T\boldsymbol{K}(T_0) = \boldsymbol{0}. \tag{41}$$

The *Riccati equation,* Equation (41), is a nonlinear matrix differential equation and is likely to cause computational difficulties. It is possible, however, to use a matrix transformation obviating the need for solving nonlinear equations, To this end, we introduce the transformation

$$\boldsymbol{K}(T_0) = \boldsymbol{E}(T_0)\boldsymbol{F}^{-1}(T_0). \tag{42}$$

We consider the matrix

$$\boldsymbol{C} = -\boldsymbol{A} + \boldsymbol{b}\boldsymbol{R}^{-1}\boldsymbol{b}^T\boldsymbol{K}(T_0), \tag{43}$$

and write the eigenvalue problem associated with $\boldsymbol{C}$ in the form

$$\boldsymbol{F}^{-1}\boldsymbol{C}\boldsymbol{F} = \boldsymbol{J}_C, \tag{44}$$

where $\boldsymbol{J}_{\mathrm{C}}$ is the matrix of eigenvalues of $\boldsymbol{C}$, assumed to be diagonal, and $\boldsymbol{F}$ is the matrix of eigenvectors. Multiplying Equation (44) on the left by $\boldsymbol{KF}$ and considering Equations (41) and (43), we obtain

$$\boldsymbol{KFF}^{-1}\boldsymbol{CF} = \boldsymbol{KCF} = \boldsymbol{KbR}^{-1}\boldsymbol{b}^T\boldsymbol{KF} - \boldsymbol{KAF} = \boldsymbol{QF} + \boldsymbol{A}^T\boldsymbol{KF} = \boldsymbol{KFJ}_C. \tag{45}$$

Moreover, multiplying Equation (44) on the left by $\boldsymbol{F}$ and considering Equation (43), we have

$$\boldsymbol{CF} = \boldsymbol{bR}^{-1}\boldsymbol{b}^T\boldsymbol{KF} - \boldsymbol{AF} = \boldsymbol{FJ}_C. \tag{46}$$

Next, introduce the transformation

$$\boldsymbol{K}(T_0)\boldsymbol{F}(T_0) = \boldsymbol{E}(T_0),$$

so that Equations (45) and (46) can be combined into

$$\begin{bmatrix} \boldsymbol{A}^T & \boldsymbol{Q} \\ \boldsymbol{bR}^{-1}\boldsymbol{b}^T & -\boldsymbol{A} \end{bmatrix} \begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{F} \end{bmatrix} = \begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{F} \end{bmatrix} \boldsymbol{J}_C, \tag{47}$$

we have to solve the eigenproblem and retain the eigenvalues with positive real parts.[25]

4.1 | The zero-order approximation of the modal gain vector

Using Equations (35) and (36), we conclude that the optimal feedback control gain matrix has the form

$$\boldsymbol{G}_0{}^T = -\boldsymbol{R}^{-1}\boldsymbol{b}^T\boldsymbol{K}(T_0), \tag{48}$$

using the transformations

$$\begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{F} \end{bmatrix} = \begin{bmatrix} (\boldsymbol{V^H})^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_1 \\ \boldsymbol{F}_1 \end{bmatrix}, \tag{49}$$

and premultiplication of Equation (47) by

$$\begin{bmatrix} \boldsymbol{U}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V^H} \end{bmatrix},$$

since $\boldsymbol{V}^H\boldsymbol{U} = I$ and $\boldsymbol{b}_1 = \boldsymbol{V}^H\boldsymbol{b}$, then Equation (47) are changed into

$$\begin{bmatrix} \boldsymbol{J}^T & \boldsymbol{U}^T\boldsymbol{QU} \\ \boldsymbol{b}_1\boldsymbol{R}^{-1}\boldsymbol{b}_1^T & -\boldsymbol{J} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_1 \\ \boldsymbol{F}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{E}_1 \\ \boldsymbol{F}_1 \end{bmatrix} \boldsymbol{J}_C. \tag{50}$$

The steady-state solution of the modal Riccati matrix Equation (41) is as follows

$$\boldsymbol{K} = \boldsymbol{EF}^{-1} = (\boldsymbol{V^H})^T\boldsymbol{E}_1\boldsymbol{F}_1^{-1}\boldsymbol{V^H}, \tag{51}$$

Equation (50) shows that if the Potter's algorithm is extended to solve Equation (41) for $\boldsymbol{K}$ of the center subspace corresponding to $2\,m$ non-semi-simple eigenvalues, only $2\,m$ eigenvalues and eigenvectors are required for computations in Equation (49).

Using Equation (48), the gain vector can be written in the form

$$\boldsymbol{G}_0{}^T = -\boldsymbol{R}^{-1}\boldsymbol{b}^T\boldsymbol{K} = -\boldsymbol{R}^{-1}\boldsymbol{b}^T(\boldsymbol{V^H})^T\boldsymbol{E}_1\boldsymbol{F}_1^{-1}\boldsymbol{V^H}, \tag{52}$$

considering Equation (26), $\boldsymbol{g}_0{}^T = \boldsymbol{G}_0{}^T\boldsymbol{U}$, the zero-order approximation of the modal gain vector in Equation (24), $\boldsymbol{g}_0{}^T$, can be deduced from Equation (52)

$$\boldsymbol{g}_0{}^T = -\boldsymbol{R}^{-1}\boldsymbol{b}_1^T\boldsymbol{K}_1, \tag{53}$$

where

$$\boldsymbol{K}_1 = \boldsymbol{E}_1\boldsymbol{F}_1^{-1}.$$

Using Equations (24) and (53), the closed-loop modal equation can be written in the form

$$D_0\xi_0 = \boldsymbol{J}\xi_0 + \boldsymbol{b}_1 z_0 = (\boldsymbol{J} + \boldsymbol{b}_1\boldsymbol{g}_0{}^T)\xi_0 = \boldsymbol{H}_1\xi_0, \tag{54}$$

where

$$\boldsymbol{H}_1 = \boldsymbol{J} + \boldsymbol{b}_1\boldsymbol{g}_0{}^T. \tag{55}$$

The control input can be deduced from Equations (16) and (17) that

$$\begin{aligned} z &= (\boldsymbol{G}_0 + \varepsilon\boldsymbol{G}_1)^T(\boldsymbol{x}_0 + \varepsilon\boldsymbol{x}_1) = (\boldsymbol{g}_0 + \varepsilon\boldsymbol{g}_1)^T(\xi_0 + \varepsilon\xi_1), \\ &= \boldsymbol{g}_0{}^T\xi_0 + \varepsilon(\boldsymbol{g}_1{}^T\xi_0 + \boldsymbol{g}_0{}^T\xi_1) \\ &= z_0 + \varepsilon z_1 \end{aligned} \tag{56}$$

where

$$\begin{aligned} z_0 &= \boldsymbol{g}_0{}^T\xi_0, \\ z_1 &= \boldsymbol{g}_1{}^T\xi_0 + \boldsymbol{g}_0{}^T\xi_1. \end{aligned} \tag{57}$$

4.2 | The first-order modification of the modal gain vector

From Equation (25), we obtain

$$\begin{aligned} D_0\xi_1 &= \boldsymbol{J}\xi_1 - D_1\xi_0 + \boldsymbol{b}_1(\boldsymbol{g}_1{}^T\xi_0 + \boldsymbol{g}_0{}^T\xi_1) + \boldsymbol{r} \\ &= (\boldsymbol{J} + \boldsymbol{b}_1\boldsymbol{g}_0{}^T)\xi_1 - D_1\xi_0 + \boldsymbol{b}_1\boldsymbol{g}_1{}^T\xi_0 + \boldsymbol{r} \\ &= \boldsymbol{H}_1\xi_1 + \boldsymbol{b}_1\boldsymbol{g}_1{}^T\xi_0 - D_1\xi_0 + \boldsymbol{r}. \end{aligned} \tag{58}$$

Solving the eigenvalue problem of $\boldsymbol{H}_1$, the Jordan matrix $\boldsymbol{J}_1$ and corresponding modal matrices $\boldsymbol{U}_1$, $\boldsymbol{V}_1$ of $\boldsymbol{H}_1$ can be obtained,

$$\boldsymbol{U}_1 = [\boldsymbol{u}_{11}, \boldsymbol{u}_{12}, \cdots, \boldsymbol{u}_{2m}], \boldsymbol{V}_1 = [\boldsymbol{v}_{11}, \boldsymbol{v}_{12}, \cdots, \boldsymbol{v}_{2m}], \tag{59}$$

where $\boldsymbol{u}_{1i}$ and $\boldsymbol{v}_{1i}$ are the ith vector of the submatrix $\boldsymbol{U}_1$ and $\boldsymbol{V}_1$ ($i = 1,2,\ldots,2m$), respectively. They satisfy the following equation,

$$\boldsymbol{V}_1^{\boldsymbol{H}}\boldsymbol{U}_1 = I.$$

Using the modal transformations

$$\xi_0 = \boldsymbol{U}_1\boldsymbol{\zeta}_0, \xi_1 = \boldsymbol{U}_1\boldsymbol{\zeta}_1, \tag{60}$$

Equations (54) and (58) are changed into

$$D_0\boldsymbol{\zeta}_0 = \boldsymbol{J}_1\boldsymbol{\zeta}_0, \tag{61}$$

and

$$D_0\boldsymbol{\zeta}_1 = \boldsymbol{J}_1\boldsymbol{\zeta}_1 - D_1\boldsymbol{\zeta}_0 + \boldsymbol{V}_1^{\boldsymbol{H}}\boldsymbol{b}_1\boldsymbol{g}_1^T\boldsymbol{U}_1\boldsymbol{\zeta}_0 + \boldsymbol{V}_1^{\boldsymbol{H}}\boldsymbol{r}(\boldsymbol{\zeta}_0), \tag{62}$$

where

$$\boldsymbol{\zeta}_0 = [\boldsymbol{\zeta}_{01}, \xi_{02}, \ldots, \xi_{02m}]^T, \boldsymbol{\zeta}_1 = [\boldsymbol{\zeta}_{11}, \xi_{12}, \ldots, \xi_{12m}]^T.$$

and $\boldsymbol{J}_1$ is the Jordan canonical form of $\boldsymbol{H}_1$.

It should be noted, if the constituent Jordan blocks of the matrix $\boldsymbol{J}$ in Equation (24) is controllable (ie, $b_{1m}^{(1)} \neq 0$ and $b_{1m}^{(2)} \neq 0$), after the feedback control design, all the modes can be stabilized, the eigenvalues of $\boldsymbol{J}_1$ may be changed into distinct ones,

$$\boldsymbol{J}_1 = \mathrm{diag}\{\lambda_{11}, \lambda_{12}, \ldots, \lambda_{12m}\}. \tag{63}$$

Hence, the solutions of (61) have the following form

$$\begin{aligned} &\zeta_{0i} = c_i(T_1)e^{\lambda_{1i}T_0}(i = 1, 2, \ldots, 2m), \\ &\boldsymbol{\zeta}_0 = [c_1(T_1)e^{\lambda_{11}T_0}, c_2(T_1)e^{\lambda_{12}T_0}, \ldots, c_{2m}(T_1)e^{\lambda_{12m}T_0}]^T, \end{aligned} \tag{64}$$

where ζ_{0i} is the ith element of the state vector $\boldsymbol{\zeta}_0$.

Equation (58) can be written in the form

$$D_0\boldsymbol{\zeta}_1 = \boldsymbol{J}_1\boldsymbol{\zeta}_1 - D_1\boldsymbol{\zeta}_0 + \boldsymbol{\mu}\boldsymbol{\zeta}_0 + \boldsymbol{r}_1(\boldsymbol{\zeta}_0), \tag{65}$$

where

$$\begin{aligned} &\boldsymbol{\mu} = \boldsymbol{V}_1^{\boldsymbol{H}}\boldsymbol{b}_1\boldsymbol{g}_1{}^T\boldsymbol{U}_1 = \boldsymbol{b}_2\boldsymbol{g}_1{}^T\boldsymbol{U}_1, \\ &\boldsymbol{b}_2 = \boldsymbol{V}_1^{\boldsymbol{H}}\boldsymbol{b}_1 = = [b_{21}, b_{22}, \ldots, b_{22m}]^T, \\ &\boldsymbol{r}_1 = \boldsymbol{V}_1^{\boldsymbol{H}}\boldsymbol{r}. \end{aligned} \tag{66}$$

the ith row matrix of $\boldsymbol{\mu}$ is

$$\boldsymbol{\mu}_i = [b_{2i}\boldsymbol{g}_1{}^T\boldsymbol{u}_{11}, b_{2i}\boldsymbol{g}_1{}^T\boldsymbol{u}_{12}, \ldots, b_{2i}\boldsymbol{g}_1{}^T\boldsymbol{u}_{1i}, \ldots, b_{2i}\boldsymbol{g}_1{}^T\boldsymbol{u}_{12m}], = [\mu_{i1}, \mu_{i2}, \ldots, \mu_{ii}, \ldots, \mu_{i2m}] \tag{67}$$

where μ_{ij} is the jth element of the ith row of the matrix $\boldsymbol{\mu}$, and $\boldsymbol{u}_{1i}$ is ith vector of the matrix $\boldsymbol{U}_1$. Hence, the ith equation of Equation (62) is

$$D_0\zeta_{1i} = \lambda_{1i}\zeta_{1i} - D_1\zeta_{0i} + \boldsymbol{\mu}_i\boldsymbol{\zeta}_0 + \boldsymbol{r}_{1i} = \lambda_{1i}\zeta_{1i} - D_1\zeta_{0i} + \mu_{ii}\zeta_{0i} + \sum_{\substack{j=1 \\ j \neq i}}^{2m} \mu_{ij}\zeta_{0j} + r_{1i} \ (i = 1, 2, \ldots, 2m), \tag{68}$$

using Equation (64), Equation (68) can be written in the form

$$D_0\zeta_{1i} = \lambda_{1i}\zeta_{1i} - (D_1c_i(T_1) - c_i(T_1)\mu_{ii})e^{\lambda_{1i}T_0} + \sum_{\substack{k=1 \\ k \neq i}}^{2m} \mu_{ik}e^{\lambda_{1k}T_0}c_k(T_1) + r_{1i}(i = 1, 2, \ldots, 2m). \tag{69}$$

In Equation (69), the modal forces r_{1i} are the function of $(\lambda_{1i})^3$ (i = 1,2,..., $2m$) relating to the nonlinear forces of the cubic displacements, and $c_i(T_1)$ can be determined by eliminating the secular terms. If internal resonance between modes is absent,

$$-D_1c_i(T_1) + c_i(T_1)\mu_{ii} = 0(i = 1, 2, \ldots, 2m). \tag{70}$$

The solutions of Equation (70) are

$$c_i(T_1) = d_i \cdot e^{\mu_{ii}T_1}(i = 1, 2, \ldots, 2m). \tag{71}$$

Substituting Equation (71) into Equation (64), solutions $\zeta_i(i = 1, 2, \ldots, 2m)$ can be obtained

$$\zeta_{0i} = d_i \cdot e^{\mu_{ii}T_1} \cdot e^{\lambda_{1i}T_0}(i = 1, 2, \ldots, 2m), \tag{72}$$

where the constants d_i can be computed by the initial condition

$$\zeta_0(0) = \boldsymbol{V}_1^{\boldsymbol{H}} V^{\boldsymbol{H}} \boldsymbol{x}_0(0). \tag{73}$$

Recalling the Equation (67), coefficients μ_{ii} in Equation (72) are given by $\mu_{ii} = b_{2i}\boldsymbol{g}_1^T\boldsymbol{u}_{1i} = b_{2i}\boldsymbol{u}_{1i}^T\boldsymbol{g}_1$ ($i = 1,2,...,2m$), they can be written in a compact form

$$\boldsymbol{P}\boldsymbol{U}_1^T\boldsymbol{g}_1 = \boldsymbol{\mu}_e, \tag{74}$$

where

$$\boldsymbol{P} = \begin{bmatrix} b_{21} & & & \\ & b_{22} & 0 & \\ & 0 & \ddots & \\ & & & b_{22m} \end{bmatrix}, \quad \boldsymbol{\mu}_e = \begin{bmatrix} \mu_{11} \\ \mu_{22} \\ \vdots \\ \mu_{2m2m} \end{bmatrix}. \tag{75}$$

In the case of $b_{2i} \neq 0$ ($i = 1,2,...,2m$), the Equation (74) can be written in the form

$$\boldsymbol{g}_1 = \boldsymbol{M}_1\boldsymbol{\mu}_e, \tag{76}$$

where

$$\boldsymbol{M}_1 = (\boldsymbol{P}\boldsymbol{U}_1^T)^{-1}, \tag{77}$$

Since $\boldsymbol{g}_1^T = \boldsymbol{G}_1^T\boldsymbol{U}$, the first-order modification of the modal gain vector in Equation (18), $\boldsymbol{G}_1$, can be deduced from Equation (75),

$$\boldsymbol{G}_1 = \boldsymbol{M}_2\boldsymbol{\mu}_e, \tag{78}$$

where

$$\boldsymbol{M}_2 = (\boldsymbol{P}\boldsymbol{U}_1^T\boldsymbol{U}^T)^{-1}, \tag{79}$$

and matrices $\boldsymbol{U}$ and $\boldsymbol{U}_1$ are given by Equations (20) and (59), respectively.

Equation (72) shows that the dynamic characteristic of the close-loop system $\boldsymbol{H}_1$ can be changed by the coefficient μ_{ii} because of the perturbations $\boldsymbol{g}_1$ of the modal gain vector. So, μ_{ii} can be assigned to obtain required dynamic properties. It is important to note Equation (78) that the gain vector $\boldsymbol{G}_1$ is a real vector, matrix $\boldsymbol{M}_2$ is a complex matrix, so only some elements $\mu_{ii}(i = 1, 2, \dots, m)$ of the vector $\boldsymbol{\mu}_e$ are selected arbitrarily, that is to satisfy the equation

$$Im(\boldsymbol{G}_1) = Im(\boldsymbol{M}_2\boldsymbol{\mu}_e) = \boldsymbol{0}. \tag{80}$$

If one wants to keep the dynamic properties of the close-loop system $\boldsymbol{H}_1$, one way is to let $\boldsymbol{\mu}_e = \boldsymbol{0}$. Therefore, the solution can be solved if the coefficient determinant of Equation (74) is not equal to zero, that is, $\det(\boldsymbol{P}\boldsymbol{U}_1^T) \neq 0$, and the constants b_{2i} ($i = 1,2,...,2m$) is not equal to zero satisfy this condition. Then

$$g_{11} = g_{12} = \dots = g_{12m} = 0. \tag{81}$$

Using Equation (60), solution ξ_0 can be obtained. Under condition (70), solutions ζ_{1i} ($i = 1,2,...,2m$) can be solved from Equation (69), then solution ξ_1 can be obtained.

The input z can be computed as follows

$$\begin{aligned} z &= z_0 + \varepsilon z_1 = \boldsymbol{g}_0{}^T\xi_0 + \varepsilon(\boldsymbol{g}_0{}^T\xi_1{}^T + \boldsymbol{g}_1{}^T\xi_0{}^T). \\ &= \boldsymbol{g}_0{}^T\boldsymbol{V}^{\boldsymbol{H}}(\boldsymbol{x}_0 + \varepsilon\boldsymbol{x}_1) + \varepsilon\boldsymbol{g}_1{}^T\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{x}_0 \\ &= \boldsymbol{g}_0{}^T\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{x} + \varepsilon\boldsymbol{g}_1{}^T\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{x}_0 \end{aligned} \tag{82}$$

Substituting Equation (82) into (10), the nonlinear control closed-loop system can be written in the form

$$\begin{aligned}\dot{\boldsymbol{x}} &= (\boldsymbol{A} + \boldsymbol{b}\boldsymbol{g}_0^T\boldsymbol{V}^{\boldsymbol{H}})\boldsymbol{x} + \varepsilon\boldsymbol{f}(\boldsymbol{x}) + \varepsilon\boldsymbol{b}\boldsymbol{g}_1^T\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{x}_0,\\ &= \boldsymbol{A}_1\boldsymbol{x} + \varepsilon\boldsymbol{f}(\boldsymbol{x}) + \varepsilon\boldsymbol{c}\boldsymbol{x}_0,\end{aligned} \tag{83}$$

where

$$\boldsymbol{A}_1 = \boldsymbol{A} + \boldsymbol{b}\boldsymbol{g}_0^T\boldsymbol{V}^{\boldsymbol{H}}, \boldsymbol{c} = \boldsymbol{b}\boldsymbol{g}_1^T\boldsymbol{V}^{\boldsymbol{H}}. \tag{84}$$

Equation (82) indicates that after the feedback control design, the state matrix $\boldsymbol{A}$ is changed into $\boldsymbol{A}_1$. The modes of $\boldsymbol{A}$ have been stabilized and $\boldsymbol{J}_1$ has been diagonalized.

5 | APPLICATION EXAMPLE

Consider the flutter problem of an airfoil in simplified formulation. The airfoil is placed by a rigid rectangular panel with two degrees of freedom, the vertical displacement h and the rotation α. It is assumed that aerodynamic lift force is proportional to the angle of attack α and the square of the velocity v of flight. The nonlinear differential equations of motion[26] are

$$\begin{cases} m\ddot{h} + s\ddot{\alpha} + \boldsymbol{K}_h h + \rho v^2 ab_0\alpha = \varepsilon\boldsymbol{K}_h h^3 = \varepsilon Q_1 \\ s\ddot{h} + J_\alpha\ddot{\alpha} + \boldsymbol{K}_\alpha\alpha + \rho v^2 ab_0 e\alpha = \varepsilon\boldsymbol{K}_\alpha\alpha^3 = \varepsilon Q_2 \end{cases}, \tag{85}$$

where m is the mass of the panel, s the static moment of the cross section area of the panel, J_α is the moment of inertia, K_h the bending stiffness, K_α the torsional stiffness, ε is a small parameter respectively, εQ_1 and εQ_2 are the nonlinear forces.

If the parameters are given as $m/(\rho abb_0^2) = 5$, $s/m = 0.25$, $J_\alpha/m = 0.5$, $e = 0.4$, $K_h/m = 0.25$, $K_\alpha/J_\alpha = 1$, and $p^2 = v^2/b_0$, then the linearized equations become

$$\boldsymbol{M}_0\ddot{\boldsymbol{q}} + [\boldsymbol{K}_0 + \boldsymbol{H}_0(p^2)]\boldsymbol{q} = 0, \tag{86}$$

where

$$\boldsymbol{M}_0 = \begin{bmatrix} 1 & 0.25 \\ 0.25 & 0.5 \end{bmatrix}, \boldsymbol{K}_0 + \boldsymbol{H}_0(p^2) = \begin{bmatrix} 0.25 & 0.2p^2 \\ 0 & 0.5 - 0.08p^2 \end{bmatrix}, \boldsymbol{q} = [h, \alpha]^T, \tag{87}$$

Assuming the parameter $p = p_c = 1.32567735$, the state matrix is

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & -\boldsymbol{M}_0^{-1}(\boldsymbol{K}_0 + \boldsymbol{H}_0) \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} 0.0 & 0.0 & -0.28571429 & -0.19632103 \\ 0.0 & 0.0 & 0.14285714 & -0.62065221 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \end{bmatrix}, \tag{88}$$

where $\boldsymbol{M}_0$ is the mass matrix, $\boldsymbol{K}_0$ the stiffness matrix, $\boldsymbol{H}_0$ the asymmetric aerodynamic matrix and the state vector is

$$\boldsymbol{x} = [\dot{h}, \dot{\alpha}, h, \alpha]^T. \tag{89}$$

The flutter of the airfoil is characterized by the conditions that if the eigenvalues of the matrix $\boldsymbol{A}$, $Re(\lambda) = 0$, $Im(\lambda) \neq 0$, which describe the critical state of the flutter, and if $Re(\lambda) > 0$, $Im(\lambda) \neq 0$, which describe the flutter occurrence, and eigenvalue is also the corresponding flutter frequency.

The eigenvalues of the matrix $\boldsymbol{A}$ can be shown to be

$$\lambda_1 = \lambda_2 = 0.67318887j, \lambda_3 = \lambda_3 = -0.67318887j, \tag{90}$$

where $j = \sqrt{-1}$, λ_1, λ_2, and λ_3, λ_4 are two pairs of two-multiple non-semi-simple eigenvalues. Because $Re(\lambda_i) = 0$, $Im(\lambda_i) \neq 0$ $(i = 1,2,3,4)$, the system is in the critical state of the flutter.

The Jordan matrix of the system is

$$\boldsymbol{J}=\begin{bmatrix}\lambda_1 & 1 & 0 & 0\\ 0 & \lambda_2 & 0 & 0\\ 0 & 0 & \lambda_3 & 1\\ 0 & 0 & 0 & \lambda_4\end{bmatrix}=\begin{bmatrix}0.67318887j & 1 & 0 & 0\\ 0 & 0.67318887j & 0 & 0\\ 0 & 0 & -0.67318887j & 1\\ 0 & 0 & 0 & -0.67318887j\end{bmatrix}. \tag{91}$$

Equation (91) shows that the system has two Jordan blocks with two-multiple non-semi-simple eigenvalue, The right and left modal matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ can be computed as follows

$$\begin{cases}\boldsymbol{U}=\begin{bmatrix}0.10545750j & -0.24929491 & -0.10545750j & -0.24929491\\ 0.08995980j & 0.51057428 & -0.08995980j & 0.51057428\\ 0.15665366 & 0.60302328j & 0.15665366 & -0.60302328j\\ 0.13363127 & -0.55993649j & 0.13363127 & 0.55993649j\end{bmatrix}\\ \boldsymbol{V}=\begin{bmatrix}3.34714013j & -0.58973916 & -3.34714013j & -0.58973916\\ 1.63428715j & 0.69134119 & -1.63428715j & 0.69134119\\ 1.66351831 & 0.39700585j & 1.66351831 & -0.39700585j\\ 1.79155103 & -0.46540319j & 1.79155103 & 0.46540319j\end{bmatrix}\end{cases}. \tag{92}$$

Using the perturbation given by Chen[26] and Deif,[27] If, $\varepsilon = p - p_c = 1.32567735 - 1.32467735 = 0.001$, we obtain the eigenvalues of the perturbed system

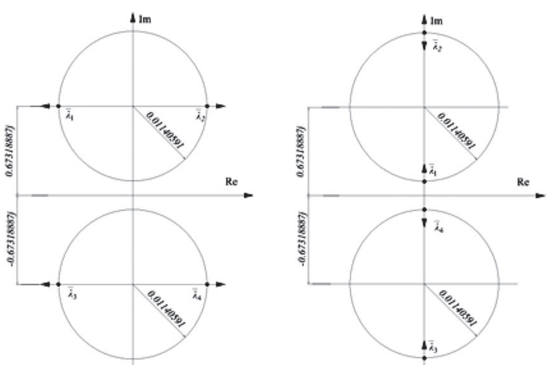
$$\begin{aligned}&\overline{\lambda}_1 = -0.01140591 + 0.67318887j, \overline{\lambda}_2 = 0.01140591 + 0.67318887j,\\ &\overline{\lambda}_3 = -0.01140591 - 0.67318887j, \overline{\lambda}_4 = 0.01140591 - 0.67318887j\end{aligned} \tag{93}$$

and if $\varepsilon = p - p_c = -0.001$, then obtain

$$\begin{aligned}&\overline{\lambda}_1 = 0.01140591e^{\frac{3\pi}{2}j} + 0.67318887j\ , \overline{\lambda}_2 = 0.01140591e^{\frac{\pi}{2}j} + 0.67318887j\ ,\\ &\overline{\lambda}_3 = 0.01140591e^{\frac{3\pi}{2}j} - 0.67318887j, \overline{\lambda}_4 = 0.01140591e^{\frac{\pi}{2}j} - 0.67318887j\ .\end{aligned} \tag{94}$$

The above results given by Equations (93) and (94), are also shown in Figure 1.

From Equations (93),(94) and Figure 1, it can be seen that when the parameter p is increased for $p > p_c$ the increment of eigenvalues diverge along horizontal line; and for $p < p_c$, the increment of eigenvalues, approach to origin along vertical line. The arrows in the Figure 1 show the movement of the eigenvalue λ, when the parameter p is changed. According to the variations of real parts of eigenvalues that can identify the bifurcation of the system is a Hopf bifurcation. Because



(A) p-p_c=0.001 (B) p-p_c=-0.001

Figure 1 Eigenvalue bifurcation at the critical point for the example. A, p-$p_c = 0.001$; B, p-$p_c = 0.001$

it results from two eigenvalues $\overline{\lambda}_2$ and $\overline{\lambda}_4$ transversely crossing the imaginary axis, as the parameter p pass through the critical value p_c. This is the mechanism of the instability of Hopf bifurcation with non-semi-simple eigenvalue.

Assume the input z_0 is applied to the rotation degree of freedom, then

$$\begin{aligned}\boldsymbol{b} &= \begin{bmatrix} \boldsymbol{M}_0^{-1}\begin{bmatrix}0\\1\end{bmatrix} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.571429 \\ 2.285714 \\ 0 \\ 0 \end{bmatrix}, \\ \boldsymbol{b}_1 &= \boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{b} = [p_1^{(1)}, p_2^{(1)}, p_1^{(2)}, p_2^{(2)}]^T \\ &= [-1.822862j, 1.917202, 1.822862j, 1.917202]^T. \end{aligned} \tag{95}$$

Now, it is evident that Equation (24) may be written in the form

$$\begin{aligned} D_0\xi_0 &= \boldsymbol{J} + \boldsymbol{b}_1\boldsymbol{g}_0^T\xi_0 \\ &= \begin{bmatrix} \boldsymbol{J}^{(1)} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}^{(2)} \end{bmatrix} + \begin{bmatrix} b_{11}^{(1)} \\ b_{12}^{(1)} \\ b_{11}^{(2)} \\ b_{12}^{(2)} \end{bmatrix}\boldsymbol{g}_0^T\xi_0 \\ &= \begin{bmatrix} 0.67318887j & 1 & 0 & 0 \\ 0 & 0.67318887j & 0 & 0 \\ 0 & 0 & -0.67318887j & 1 \\ 0 & 0 & 0 & -0.67318887j \end{bmatrix}\xi_0 + \begin{bmatrix} -1.822862j \\ 1.917202 \\ 1.822862j \\ 1.917202 \end{bmatrix}\boldsymbol{g}_0^T\xi_0. \end{aligned} \tag{96}$$

According to the control theory, it is evident that the $\boldsymbol{J}^{(1)}$ and $\boldsymbol{J}^{(2)}$ blocks in this example are therefore controllable by the input variable if and only if $b_{12}^{(1)} \neq 0$ and $b_{12}^{(2)} \neq 0$. Since $b_{12}^{(1)} = b_{12}^{(2)} = 1.917202 \neq 0$, the $\boldsymbol{J}^{(1)}$ and $\boldsymbol{J}^{(2)}$ blocks are controllable, respectively. Therefore, the system (96) is controllable.

If $\boldsymbol{H}$, $\boldsymbol{Q}$, and $\boldsymbol{R}$, in Equation (27) are given by

$$\boldsymbol{H} = \boldsymbol{0}, \boldsymbol{Q} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix}, \boldsymbol{R} = 1,$$

and $\boldsymbol{b}_1$ is given by Equation (95), the eigen problem (50) has the following form

$$\begin{bmatrix} 0.67318887j & 0 & 0 & 0 & 0.25 & 0 & 0 & 0 \\ 1 & 0.67318887j & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & -0.67318887j & 0 & 0 & 0 & 1 & 0.25 \\ 0 & 0 & 1 & -0.67318887j & 0 & 0 & 0.25 & 0.5 \\ -3.32282579 & -3.49479506j & 3.32282579 & -3.49479506j & -0.67318887j & -1 & 0 & 0 \\ -3.49479506j & 3.67566442 & 3.49479506j & 3.67566442 & 0 & -0.67318887j & 0 & 0 \\ 3.32282579 & 3.49479506j & -3.32282579 & 3.49479506j & 0 & 0 & 0.67318887j & -1 \\ -3.49479506j & 3.67566442 & 3.49479506j & 3.67566442 & 0 & 0 & 0 & 0.67318887j \end{bmatrix}\begin{bmatrix} \boldsymbol{E}_1 \\ \boldsymbol{F}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{E}_1 \\ \boldsymbol{F}_1 \end{bmatrix}\boldsymbol{S}, \tag{97}$$

solving the eigenproblem, we obtain the eigenvalues with positive real part as follow,

$$\begin{aligned} S_1 &= 1.08434335 + 0.66358309j, \quad S_2 = 1.08434335 - 0.66358309j, \\ S_3 &= 0.31052487 + 0.62554888j, \quad S_4 = 0.31052487 - 0.62554888j, \end{aligned}$$

and corresponding eigenvector matrix is

$$\begin{bmatrix} \boldsymbol{E}_1 \\ \boldsymbol{F}_1 \end{bmatrix} = \begin{bmatrix} 0.02212830-0.00328729j & 0.01156367-0.00594270j & 0.12492398+0.05062977j & 0.01943460+0.01813315j \\ -0.09792463-0.15306986j & -0.00600158-0.04993300j & -0.07108757+0.15932534j & -0.01916680+0.05891393j \\ 0.01156367+0.00594270j & 0.02212830+0.00328729j & 0.01943460-0.01813315j & 0.12492398-0.05062979j \\ -0.00600158+0.04993300j & -0.09792463+0.15306986j & -0.01916680-0.05891393j & -0.07108757-0.15932534j \\ 0.14149796+0.15684187j & 0.71393538 & 0.01755020-0.00813480j & 0.92578591 \\ -0.33931716+0.03464951j & -0.37860447+0.38701255j & -0.01562026+0.06666621j & -0.28787403+0.04282883j \\ 0.71393538 & 0.14149796-0.15684187j & 0.92578591 & 0.01755020+0.00813480j \\ -0.37860447-0.38701255j & -0.33931716-0.03464951j & -0.28787403-0.04282883j & -0.01562026-0.06666621j \end{bmatrix}. \quad (98)$$

The matrix $\boldsymbol{K}_1$ of Equation (53) is

$$\boldsymbol{K}_1 = \boldsymbol{E}_1\boldsymbol{F}_1^{-1} = \begin{bmatrix} -0.03763394+0.02528015j & -0.24369401-0.06729195j & 0.23394340 & 0.30523353-0.27146528j \\ -0.24369401-0.06729195j & -0.66116362-0.82756824j & 0.30523353+0.27146528j & 1.43931905 \\ 0.23394340 & 0.305233530+0.27146528j & -0.03763394-0.02528015j & -0.24369401+0.06729195j \\ 0.30523353-0.27146528j & 1.43931905 & -0.24369401+0.06729195j & -0.66116362+0.82756824j \end{bmatrix}, \quad (99)$$

and the corresponding zero-order approximation $\boldsymbol{g}_0$ of the modal gain is

$$\boldsymbol{g}_0^T = -\boldsymbol{R}^{-1}\boldsymbol{b}_1^T\boldsymbol{K}_1 = \begin{bmatrix} -0.16406592+0.15441811j \\ -0.87437365+0.58599655j \\ -0.16406592-0.15441811j \\ -0.87437365-0.58599655j \end{bmatrix}^T. \quad (100)$$

Using Equation (55), one has the state matrix $\boldsymbol{H}_1$ of the closed-loop system

$$\boldsymbol{H}_1 = \begin{bmatrix} 0.28148291+0.97225840j & 2.06819083+1.59386249j & -0.28148291+0.29906954j & -1.06819083+1.59386249j \\ -0.31454755+0.29605075j & -1.67635113+1.79666276j & -0.31454756-0.29605075j & -1.67635113-1.12347389j \\ -0.28148291-0.29906954j & -1.06819083-1.59386249j & 0.28148291-0.97225840j & 2.06819083-1.59386249j \\ -0.31454756+0.29605075j & -1.67635113+1.12347390j & -0.31454755-0.29605075j & -1.67635113-1.79666276j \end{bmatrix}, \quad (101)$$

The eigenvalues of this matrix is

$$\lambda_{11} = -1.08434335+0.66358309j, \lambda_{12} = -0.31052487+0.62554888j,$$
$$\lambda_{13} = -1.08434335-0.66358309j, \lambda_{14} = -0.31052487-0.62554888j. \quad (102)$$

and the modal matrix is

$$\boldsymbol{U}_1 = \begin{bmatrix} 0.727226344 & 0.95141602 & 0.14413215+0.15976172j & 0.01803607-0.00836001j \\ -0.38565275+0.39421736j & -0.29584374+0.04401454j & -0.34563405+0.03529456j & -0.016052670+0.06851185j \\ 0.14413215-0.15976172j & 0.01803607+0.00836001j & 0.72722634 & 0.95141602 \\ -0.345634045-0.03529456j & -0.01605270-0.06851185j & -0.38565275-0.39421736j & -0.29584374-0.04401454j \end{bmatrix}.$$

$$\boldsymbol{V}_1 = \begin{bmatrix} -0.91795893+1.18765428j & 1.58979930-0.77126931j & 0.93044991+0.16889772j & -0.39606704-0.12628539j \\ -2.46471636+5.08873777j & 1.36175703-3.11276948j & 4.15307370-0.41584247j & -1.94502948+0.03179959j \\ 0.93044991-0.16889772j & -0.39606704+0.12628539j & -0.91795893-1.18765428j & 1.58979930+0.77126931j \\ 4.15307370+0.41584247j & -1.94502948-0.03179959j & -2.46471636-5.08873777j & 1.36175703+3.11276948j \end{bmatrix}$$

(103)

Recalling that only some elements $\mu_{ii}(i = 1, 2, 3, 4)$ of the vector $\boldsymbol{\mu}_e$ are selected arbitrarily, then some elements of $\boldsymbol{\mu}_e$ are assigned

$$\mu_{11} = -1, \mu_{22} = -3, \mu_{33} = a_3 + b_3 j, \mu_{44} = a_4 + b_4 j, \tag{104}$$

that is,

$$\boldsymbol{\mu}_e = [\mu_{11}, \mu_{22}, \mu_{33}, \mu_{44}\ ]^T = [-1, -3, a_3 + b_3 j, a_4 + b_4 j\ ]^T. \tag{105}$$

Using Equation (79), the matrix $\boldsymbol{M}_2$ is

$$\boldsymbol{M}_2 = \begin{bmatrix} -1.37943963 - 0.62391557j & -3.08776067 - 2.05381912j & -1.37943963 + 0.623915565j & -3.08776067 + 2.05381912j \\ 0.09264009 - 0.15597889j & -0.33444017 - 0.51345478j & 0.09264009 + 0.15597889j & -0.33444017 + 0.51345478j \\ -0.15521689 - 0.23883412j & 0.17856780 - 1.29378331j & -0.15521689 + 0.23883412j & 0.17856780 + 1.29378331j \\ 0.70730525 + 0.23060907j & 1.12929701 - 0.04976819j & 0.70730525 - 0.23060907j & 1.12929701 + 0.04976819j \end{bmatrix}, \tag{106}$$

and substituting Equations (105) and (106) into Equation (80),

$$\boldsymbol{Im}\left(\begin{bmatrix} -1.37943963 - 0.62391557j & -3.08776067 - 2.05381912j & -1.37943963 + 0.623915565j & -3.08776067 + 2.05381912j \\ 0.09264009 - 0.15597889j & -0.33444017 - 0.51345478j & 0.09264009 + 0.15597889j & -0.33444017 + 0.51345478j \\ -0.15521689 - 0.23883412j & 0.17856780 - 1.29378331j & -0.15521689 + 0.23883412j & 0.17856780 + 1.29378331j \\ 0.70730525 + 0.23060907j & 1.12929701 - 0.04976819j & 0.70730525 - 0.23060907j & 1.12929701 + 0.04976819j \end{bmatrix}\begin{bmatrix} -1 \\ -3 \\ a_3 + b_3 j \\ a_4 + b_4 j \end{bmatrix}\right) = \boldsymbol{0}, \tag{107}$$

then the constants a_3, b_3, a_4, and b_4 can be obtained, $a_3 = -1, b_3 = 0, a_4 = -3, b_4 = 0$, that is,

$$\boldsymbol{\mu}_e = [-1, -3, -1, -3\ ]^T. \tag{108}$$

Now, it is evident that Equation (72) may be written in the form

$$\begin{aligned} \zeta_{01} &= d_1 \cdot e^{\mu_{11} T_1} \cdot e^{\lambda_{11} T_0} = d_1 \cdot e^{(\mu_{11}\varepsilon + \lambda_{11})t} \\ \zeta_{02} &= d_2 \cdot e^{\mu_{22} T_1} \cdot e^{\lambda_{22} T_0} = d_2 \cdot e^{(\mu_{22}\varepsilon + \lambda_{22})t} \\ \zeta_{03} &= d_3 \cdot e^{\mu_{33} T_1} \cdot e^{\lambda_{33} T_0} = d_3 \cdot e^{(\mu_{33}\varepsilon + \lambda_{33})t} \\ \zeta_{04} &= d_4 \cdot e^{\mu_{44} T_1} \cdot e^{\lambda_{44} T_0} = d_4 \cdot e^{(\mu_{44}\varepsilon + \lambda_{44})t}, \end{aligned} \tag{109}$$

assuming the initial conditions are given by $\boldsymbol{x} = [\dot{h}(0), \dot{\alpha}(0), h(0), \alpha(0)]^T = [1, 0, 0, 0]^T$, use Equation (73), the constants d_1, d_2, d_3, and d_4 can be obtained

$$\begin{aligned} &d_1 = -5.536260 + 9.433150j, \quad d_2 = 3.348220 - 8.501448j, \\ &d_3 = -5.536260 - 9.433150j\ , d_4 = 3.348220 + 8.501448j. \end{aligned} \tag{110}$$

Assuming $\varepsilon = 0.1$, substituting Equations (102), (108), and (110) into Equation (109), solutions $\zeta_{0i}(i = 1, 2, 3, 4)$ can be obtained

$$\boldsymbol{\zeta}_0 = \begin{bmatrix} \zeta_{01} \\ \zeta_{02} \\ \zeta_{03} \\ \zeta_{04} \end{bmatrix} = \begin{bmatrix} (-5.536260 + 9.433150j)\ e^{(-1.18434335+0.66358309j)t} \\ (3.348220\text{-}8.501448j)e^{(-0.61052487+0.62554888j)t} \\ (-5.536260\text{-}9.433150j)\ e^{(-1.18434335-0.66358309j)t} \\ (3.348220 + 8.501448j)e^{(-0.61052487-0.62554888j)t} \end{bmatrix}. \tag{111}$$

It is evident that vector $\boldsymbol{x}_0$ may be written in the form

$$\boldsymbol{x}_0 = \boldsymbol{U}\boldsymbol{U}_1\boldsymbol{\zeta}_0, \tag{112}$$

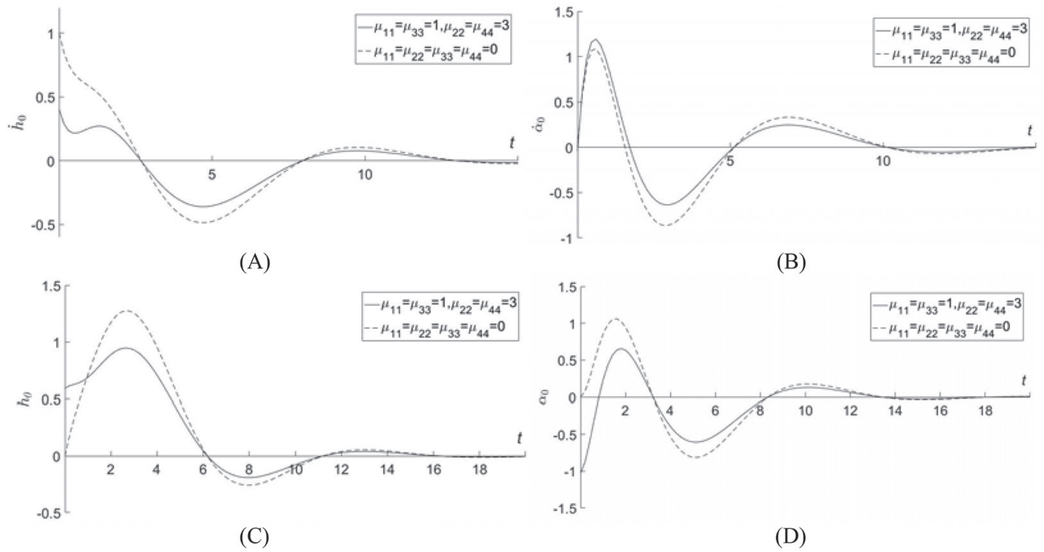


Figure 2 Zero-order solutions $\boldsymbol{x}_0$ of nonlinear aeroelastic system

then substituting Equations (92) and (103) into Equation (112), the solutions, $\boldsymbol{x}_0 = [\dot{h}_0, \dot{\alpha}_0, h_0, \alpha_0]^T$, can be obtained

$$\begin{aligned}\dot{h}_0 &= e^{-1.18434335t}(-1.304045\cos 0.663583t - 3.431456\sin 0.663583t).\\ &\quad + e^{-0.610524t}(2.304045\cos 0.625548t + 0.636997\sin 0.625548t)\\ \dot{\alpha}_0 &= e^{-1.18434335t}(-0.153651\cos 0.663583t + 9.925293\sin 0.663583t)\\ &\quad + e^{-0.610524t}(0.153651\cos 0.625548t - 3.173377\sin 0.625548t)\\ h_0 &= e^{-1.18434335t}(2.283887\cos 0.663583t + 1.766881\sin 0.663583t)\\ &\quad + e^{-0.610524t}(-2.283887\cos 0.625548t + 2.549507\sin 0.625548t)\\ \alpha_0 &= e^{-1.18434335t}(-3.972202\cos 0.663583t - 6.722416\sin 0.663583t)\\ &\quad + e^{-0.610524t}(3.972202\cos 0.625548t + 2.217442\sin 0.625548t).\end{aligned} \tag{113}$$

The solutions are plotted in Figure 2 in which curves (a), (b), (c), and (d) are in amplitude-time plane.

Figure 2 indicates that, in the design of optimal control systems, the response of the systems approaches to zero more quickly when vector $\boldsymbol{\mu}_e$ is assigned $[-1, -3, -1, -3\,]^T$ than it is assigned $[0, 0, 0, 0\,]^T$, and from Equation (109) and Figure 2, it can be seen that, by setting the value of the coefficient vector $\boldsymbol{\mu}_e$, the convergence speed of optimal control of nonlinear aeroelastic system can be adjusted.

Substituting Equation (108) into Equation (76), the first-order modification gain vector, $\boldsymbol{g}_1$, can be obtained,

$$\boldsymbol{g}_1 = \boldsymbol{M}_2\begin{bmatrix}\mu_{11}\\ \mu_{22}\\ \mu_{33}\\ \mu_{44}\end{bmatrix} = \begin{bmatrix}-0.12137018 + 0.24085576j\\ -0.43764126 + 0.41272152j\\ -0.12137018 - 0.24085576j\\ -0.43764126 - 0.41272152j\end{bmatrix}. \tag{114}$$

Using equation $\boldsymbol{g}_1^T = \boldsymbol{G}_1^T\boldsymbol{U}$, the gain vector, $\boldsymbol{G}_1$, can be obtained

$$\boldsymbol{G}_1 = \begin{bmatrix}2.12854433\\ 0.18213608\\ -0.07609731\\ -0.81903926\end{bmatrix}. \tag{115}$$

It is evident from Equation (84) that the state matrix $\boldsymbol{A_1}$ and $\boldsymbol{c}$

$$\boldsymbol{A}_1 = \boldsymbol{A} + \boldsymbol{b}\boldsymbol{g}_0^T\boldsymbol{V}^{\boldsymbol{H}} = \begin{bmatrix} -1.18001309 & 0.40243084 & -0.23967702 & 0.45128226 \\ 4.72005235 & -1.60972335 & -0.04129193 & -3.21106539 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

$$\boldsymbol{c} = \boldsymbol{b}\boldsymbol{g}_1^T\boldsymbol{V}^{\boldsymbol{H}} = \begin{bmatrix} -1.21631105 & -0.10407776 & 0.04348417 & 0.46802243 \\ 4.86524419 & 0.41631105 & -0.17393670 & -1.87208973 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{116}$$

It is not difficult to verify that the new state matrix $\boldsymbol{A_1}$ has the eigenvalues given by Equation (102), and it can be seen that after the optimal feedback control design, the system is asymptotically stable.

6 | CONCLUSIONS

The optimal control of nonlinear aeroelastic system with non-semi-simple eigenvalues at Hopf bifurcation points was developed. The method of the multiple scales and the linear quadratic regulator method were used to design the controller of the nonlinear systems. The modal control equations of nonlinear system with non-semi-simple eigenvalues were developed. The Potter's algorithm is extended to deal with the optimal control based on the modal control equations. The first-order approximation of the control of the nonlinear system was presented, which include the gain vectors $\boldsymbol{g_0}$ and $\boldsymbol{g_1}$, and inputs z_0 and z_1. Thus, the convergence speed of this nonlinear system be adjusted by set the appropriate first-order gain vector; an application example, the flutter problem of an airfoil in simplified formulation was given, from the example, it can be seen that the system with two Jordan blocks with two-multiple non-semi-simple eigenvalues has two Hopf bifurcation points. In the case, after the modal optimal control, the eigenvalues of the state matrix are as required, and the system is asymptotically stable. The results show that the present method is effective and validity for the control of nonlinear system at the Hopf bifurcation points.

In this paper, preliminary achievements have been attained on the optimal control of nonlinear systems with non-semi-simple eigenvalues at Hopf bifurcation points; however, there are many aspects to be studied in the control of critical points for nonlinear systems with nonsingle eigenvalues, such as the quantitative measures of modal controllability and observability, and feedback control for the nonlinear systems at the critical point and so on.

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LIST OF SYMBOLS

$\boldsymbol{M_0}$	mass matrix
$\boldsymbol{K_0}$	stiffness matrix
$\boldsymbol{H_0}$	asymmetric aerodynamic matrix
$\boldsymbol{Q_0}(\boldsymbol{q})$	the nonlinear elastic force
q	displacement vector
$\ddot{\boldsymbol{q}}$	acceleration vector
x	state vector
$\dot{\boldsymbol{x}}$	speed vector
$\boldsymbol{A}$	state matrix
$\boldsymbol{J}$	Jordan canonical form of the state matrix $\boldsymbol{A}$
$\boldsymbol{A}^H$	conjugate transpose of the state matrix $\boldsymbol{A}$
$\boldsymbol{z}$	control input
$\boldsymbol{b}, \boldsymbol{b}_1$	actuator distribution matrix
$\boldsymbol{G}$	modal gain vector.
$\boldsymbol{G}_0, \boldsymbol{g}_0$	zero-order of the modal gain vector
$\boldsymbol{G}_1, \boldsymbol{g}_1$	first-order of the modal gain vector.
$\boldsymbol{U}$	right modal matrices of the state matrix $\boldsymbol{A}$.
$\boldsymbol{V}$	left modal matrices of the state matrix $\boldsymbol{A}$.

$\boldsymbol{\xi}$	generalized coordinates vector,
$\boldsymbol{J}_0$	quadratic performance measure
$\mathfrak{R}$	Hamiltonian
$\boldsymbol{p}$	costate vector
$\boldsymbol{K}$	Riccati matrix

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