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Uncertainty propagation in the calibration equations for NTC thermistors

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Abstract

The uncertainty propagation problem is quite important for temperature measurements, since we rely so much on the sensors and calibration equations. Although uncertainty propagation for platinum resistance or radiation thermometers is well known, there have been few publications concerning negative temperature coefficient (NTC) thermistors. Insight into the propagation characteristics of uncertainty that develop when equations are determined using the Lagrange interpolation or least-squares fitting method is presented here with respect to several of the most common equations used in NTC thermistor calibration. Within this work, analytical expressions of the propagated uncertainties for both fitting methods are derived for the uncertainties in the measured temperature and resistance at each calibration point. High-precision calibration of an NTC thermistor in a precision water bath was performed by means of the comparison method. Results show that, for both fitting methods, the propagated uncertainty is flat in the interpolation region but rises rapidly beyond the calibration range. Also, for temperatures interpolated between calibration points, the propagated uncertainty is generally no greater than that associated with the calibration points. For least-squares fitting, the propagated uncertainty is significantly reduced by increasing the number of calibration points and can be well kept below the uncertainty of the calibration points.

Keywords: uncertainty propagation, NTC thermistor, calibration equation, Lagrange interpolation, least-squares

(Some figures may appear in colour only in the online journal)

1. Introduction

Negative temperature coefficient (NTC) thermistors are widely used in spacecraft as temperature sensors because of their small size, good reliability, short time constant, and insensitivity to mechanical shocks or vibrations. In some large space optical remote sensors, a precision of below 50 mK in temperature control for the primary mirror needs to be maintained during operations within a narrow (5 K) temperature range of 290.15 K–295.15 K. The NTC thermistors used in space optical remote sensors include the high-precision MF501 NTC thermistors[4]. Their accuracy in temperature measurements strongly depends on the resistance–temperature calibration equation used [1–3]. Generally, the coefficients of these equations are mainly determined using interpolation or curve fitting methods for a set of calibration points. The interpolation method is always regarded as providing exact fittings wherever the number of coefficients in the calibration equation is equal to the number of calibration points. In contrast, the least-squares method provides an approximate fit—as, mathematically, this method minimizes the sum of squares of all the residual errors. Nevertheless, regardless of

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[4] Certain commercial equipment, instruments or sensors are identified in this paper in order to adequately specify the experimental procedure. The MF501 NTC thermistor is provided by Chengdu Hongming Electronics Co., LTD.

the method used, uncertainties in the calibration points will propagate into subsequent measurements. Only the propagation of uncertainty for the calibration equation is accurately evaluated; nonetheless, reliable measurement results can be obtained.

In a recent paper [4], White gave a comparison of the advantages and disadvantages of several common interpolation equations with respect to the propagation of uncertainty. An unconventional method, exploiting the linear dependence of the interpolations, was provided. However, the propagated uncertainty given by these interpolation equations was assessed based only on setting the uncertainties of independent variables to zero, and no experiment was performed. In a previous paper [5], the same author gave a general introduction to the typical performance characteristics and sources of uncertainty for the calibration of NTC thermistors. Based on the polynomial interpolation method, a total uncertainty in the measured temperature over the measurement range was provided. Results show that this method is very useful when calibration equations are determined by Lagrange interpolation. Alongside this, White and Saunders [6] proposed a method, based on the theory of interpolation, to solve the problem of propagation of uncertainty for linear interpolation equations. The effects of correlation on the propagation of uncertainty and the properties of interpolation errors were discussed within a theoretical framework presented in their work. Several examples using this new method to better illustrate the calculation of propagation of uncertainties were described. However, only a numerical simulation was performed. The sources of measurement uncertainty for two types of thermometer using linear, polynomial, and power calibration equations were evaluated by Chen [7]. Results showed that the predicted uncertainty from the calibration equations was the main source for the combined uncertainty. Saunders [8, 9] provided an algebraic solution to this problem for a radiation thermometer calibrated with a modified form of the non-linear Sakuma–Hattori equation. Seven NTC thermistors were repeatedly calibrated to develop and validate a new uncertainty estimate for thermistor calibration over the temperature range of 223.15 K–363.15 K by NIST [10]. However, uncertainty evaluations were made only for several specified temperature points. For subsequent measurements, the user was left to interpolate between these points. Other studies related to the propagation of uncertainties can be found in Roberts *et al* [11], White and Saunders [12, 13], Malengo *et al* [14], and Lira *et al* [15]. Their work provides useful information on uncertainty propagation for standard platinum resistance thermometers (SPRTs) and radiation thermometers. The characteristics of NTC thermistors are quite different from those of SPRTs and radiation thermometers, yet the high-precision NTC thermistor calibrations are widely used in space or near-space projects. Therefore, any uncertainty propagation problems arising in the NTC thermistor calibration equations need to be conclusively solved; there have, however, been few publications concerning this topic. Besides, discussions of the issues involved with the uncertainty propagation in calibration equations for NTC thermistors are dispersed amongst several publications. There is no systematic level of research in this area—and, at present, the uncertainty determination of a NTC thermistor is mostly limited to a specified calibration point or the usable fixed point. What is worse, according to the review of previous research, no experiment concerning the propagated uncertainty of NTC thermistor calibration using the least-squares fitting method has been pursued. This paper provides an overview of the estimates of the propagation of uncertainties for several common NTC thermistor calibration equations determined using the Lagrange interpolation and least-squares fitting methods.

This paper is organized as follows. Section 2 presents the calibration equations for thermistors, the analytic expressions for the propagation of uncertainties for both Lagrange interpolation and least-squares fitting, and the experimental set-up of the thermistor calibration system. In section 3, the calibration results for the NTC thermistor are given, and the propagated uncertainty and interpolation residual are discussed. Conclusions are given in section 4.

2. Methods and materials

2.1. Thermistor calibration equations

Several common thermistor calibration equations are presented that describe the resistance–temperature characteristic of the NTC thermistor. The most popular and widely used equation relating thermistor resistance R to temperature T (in Kelvin) is the two-parameter exponential equation [5]

$$R = R_{T_0} \mathrm{e}^{\beta\left(\frac{1}{T} - \frac{1}{T_0}\right)}, \tag{1}$$

where R_{T_0} is the thermistor resistance at the reference temperature T_0, usually 298.15 K (25 °C). The parameter β is a characteristic of the thermistor material, with typical values in the range 2000 K–6000 K. More generally, for temperature measurements determined by NTC thermistors, $1/T$ is assumed to be a polynomial in ln$R$ or vice versa [1, 5],

$$\frac{1}{T} = \sum_{i=1}^{n} A_i (\ln R)^{i-1}, \tag{2}$$

where A_i are thermistor-dependent coefficients. Low values of n yield

(a) Basic equation ($n = 2$)

$$\frac{1}{T} = A_1 + A_2 \ln R, \tag{3}$$

(b) Hoge-1 equation ($n = 3$)

$$\frac{1}{T} = A_1 + A_2 \ln R + A_3 (\ln R)^2, \tag{4}$$

(c) Hoge-2 equation ($n = 4$)

$$\frac{1}{T} = A_1 + A_2 \ln R + A_3 (\ln R)^2 + A_4 (\ln R)^3, \tag{5}$$

(d) Hoge-3 equation ($n = 5$)

$$\frac{1}{T} = A_1 + A_2 \ln R + A_3 (\ln R)^2 + A_4 (\ln R)^3 + A_5 (\ln R_T)^4. \quad (6)$$

One may wonder why the Steinhart–Hart [16] equation, which is recommended by most manufacturers, is nevertheless omitted from our selected calibration equations. In several previous studies [1, 2, 5], the authors found that the Steinhart–Hart equation always results in poor performance of the characteristics of NTC thermistors compared to the complete three-term equation; the reasons why the use of the Steinhart–Hart equation should be strongly discouraged have been discussed in detail in a recent paper [17].

2.2. Propagation of uncertainty

The formula used in estimating the propagation of uncertainty is generally expressed as [18]

$$u_y^2 = \sum_{i=1}^{n} \left(\frac{\partial z}{\partial x_i}\right)^2 u_{x_i}^2 + \sum_{i=1}^{n} \sum_{i=1, i\neq j}^{n} \frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j} r\left(x_i, x_j\right) u_{x_i} u_{x_j}, \quad (7)$$

where $z = z\left(x_1, x_2, \cdots, x_n\right)$ is a function of the input variables $x_1, x_2, \cdots, x_n$, u_{x_i} are the associated standard uncertainties, and $r\left(x_i, x_j\right)$ denotes the correlation coefficients. Consider a general calibration equation determined by the m pairs of calibration points (x_i, y_i):

$$\widehat{y} = \widehat{y}\left(x; x_1, x_2, \cdots x_m; y_1, y_2, \cdots y_m\right). \quad (8)$$

Based on equation (7), the propagation of error of $\widehat{y}$ can be obtained by directly differentiating equation (8):

$$\mathrm{d}\widehat{y} = \sum_{i=1}^{m} \frac{\partial \widehat{y}}{\partial y_i}\mathrm{d}y_i + \sum_{i=1}^{m} \frac{\partial \widehat{y}}{\partial x_i}\mathrm{d}x_i + \frac{\partial \widehat{y}}{\partial x}\mathrm{d}x. \quad (9)$$

Regardless of the calibration equation, whether obtained by the Lagrange interpolation or least-squares fitting method, $\widehat{y}$ can be expressed in a form with y_i $(i = 1,\ldots,m)$ as coefficients [6],

$$\widehat{y} = \sum_{i=1}^{m} y_i f_i\left(x\right), \quad (10)$$

where $f_i\left(x\right) = f\left(x, x_1, x_2, \cdots, x_m\right)$. The propagation of error of $\widehat{y}$ can then be expressed as

$$\mathrm{d}\widehat{y} = \sum_{i=1}^{m} f_i(x)\mathrm{d}y_i - \sum_{i=1}^{m} f_i(x)\left(\left.\frac{\partial \widehat{y}}{\partial x}\right|_{x=x_i}\right)\mathrm{d}x_i + \frac{\partial \widehat{y}}{\partial x}\mathrm{d}x. \quad (11)$$

The factors $f_i(x)$, $f_i(x)\left(\left.\frac{\partial \widehat{y}}{\partial x}\right|_{x=x_i}\right)$, and $\frac{\partial \widehat{y}}{\partial x}$ refer to the sensitivity coefficients of the y_i variables, the x_i variables, and the x variable respectively. Hence, the propagation of uncertainty is

$$u_{\widehat{y}}^2 = \sum_{i=1}^{m} f_i^2(x) u_{y_i}^2 + \sum_{i=1}^{m} f_i^2(x)\left(\left.\frac{\partial \widehat{y}}{\partial x}\right|_{x=x_i}\right)^2 u_{x_i}^2 + \left(\frac{\partial \widehat{y}}{\partial x}\right)^2 u_x^2. \quad (12)$$

When the dependent variable is measured to estimate the value of the independent variable—as, for example regarding NTC thermistors, by treating $\ln R$ as a function of $1/T$— then the expression of the propagation of uncertainty for u_x is

$$u_x^2 = \left[\sum_{i=1}^{m} f_i^2(x) u_{y_i}^2 + \sum_{i=1}^{m} f_i^2(x)\left(\left.\frac{\partial \widehat{y}}{\partial x}\right|_{x=x_i}\right)^2 u_{x_i}^2 + u_{\widehat{y}}^2\right]\left(\frac{\partial \widehat{y}}{\partial x}\right)^{-2}. \quad (13)$$

2.2.1. Lagrange interpolation. To analyse the propagation of uncertainty for a NTC thermistor equation calibrated using the Lagrange interpolation method with m calibration points, we rewrite equation (2) in the Lagrange interpolation form [5, 6]

$$\frac{1}{T} = \sum_{i=1}^{m} \frac{1}{T_i} l_i(\ln R), \quad (14)$$

where

$$l_i(\ln R) = \prod_{j=1, j\neq i}^{m} \frac{(\ln R - \ln R_j)}{(\ln R_i - \ln R_j)}. \quad (15)$$

Substituting equation (14) into equation (11) yields the equation for the propagation of error

$$\mathrm{d}T \approx \sum_{i=1}^{m} l_i(\ln R)\left[\frac{T^2}{T_i^2}\mathrm{d}T_i + \frac{T^2}{\beta}\frac{\mathrm{d}R_i}{R_i}\right] - \frac{T^2}{\beta}\frac{\mathrm{d}R}{R}. \quad (16)$$

For a narrow temperature range (e.g. 273.15 K–333.15 K), β can be treated as a constant; then, according to equation (1),

$$l_i\left(\ln R\right) \approx \frac{T_i^{m-1}}{T^{m-1}} l_i\left(T\right), \quad (17)$$

where

$$l_i(T) = \prod_{j=1, j\neq i}^{m} \frac{(T - T_j)}{(T_i - T_j)}. \quad (18)$$

By substituting equation (17) into equation (16), the propagation of uncertainty for a NTC thermistor calibrated with m points is obtained by calculating the sum of the squares of each of the terms in equation (16):

$$u_T^2 = \sum_{i=1}^{m}\left[l_i^2(T)\left(\frac{T}{T_i}\right)^{6-2m}\left(u_{T_i}^2 + \frac{T_i^4}{\beta^2}\frac{u_{R_i}^2}{R_i^2}\right)\right] + \frac{T^4}{\beta^2}\frac{u_R^2}{R^2}. \quad (19)$$

2.2.2. Least-squares fitting. For least-squares fitting, we assume that the calibration equation (2) is determined by m ($m > n$) pairs of (x_i, y_i) calibration points and re-express it as

$$T = T(R; A_1, A_2, \cdots, A_n), \quad (20)$$

where the coefficients of A_i are determined by minimizing the sum of squares of all the residual errors,

$$\chi^2 = \sum_{i=1}^{m} \left[T_i - T(R_i)\right]^2, \quad (21)$$

obtained by solving equations that set each of the derivatives $\frac{\partial \chi^2}{\partial A_i}$ to zero, i.e.

Table 1. Resistance–temperature measurement data for the MF501 NTC thermistor. Calibration points marked with symbol '⋆' were used for both Lagrange interpolation and least-squares fitting.

		Lagrange interpolation			Least-squares fitting		
T/K	R/Ω	2-points	3-points	4-points	5-points	6-points	11-points
278.2574	13 164.11	▲	▲	⋆	⋆	⋆	▲
283.3417	10 162.63						▲
288.2827	7966.33				⋆	⋆	▲
293.1597	6311.24			⋆			▲
298.0455	5034.14					⋆	▲
302.9663	4037.07		▲		⋆		▲
307.9471	3251.18					⋆	▲
312.9821	2629.81			⋆			▲
318.0535	2138.22				⋆	⋆	▲
323.1317	1749.18						▲
328.1941	1440.67	▲	▲	⋆	⋆	⋆	▲

$$\frac{\partial \chi^2}{\partial A_1} = -2\sum_{i=1}^{m}\left[T_i - T(R_i)\right]\left.\frac{\partial T}{\partial A_1}\right|_{R=R_i} = 0$$
$$\frac{\partial \chi^2}{\partial A_2} = -2\sum_{i=1}^{m}\left[T_i - T(R_i)\right]\left.\frac{\partial T}{\partial A_2}\right|_{R=R_i} = 0$$
$$\vdots$$
$$\frac{\partial \chi^2}{\partial A_n} = -2\sum_{i=1}^{m}\left[T_i - T(R_i)\right]\left.\frac{\partial T}{\partial A_n}\right|_{R=R_i} = 0. \quad (22)$$

Hence, the coefficients A_i can be expressed in terms of the m pairs (R_i, T_i)

$$A_1 = A_1(R_1, R_2, \cdots, R_m; T_1, T_2, \cdots, T_m)$$
$$A_2 = A_2(R_1, R_2, \cdots, R_m; T_1, T_2, \cdots, T_m)$$
$$\vdots$$
$$A_n = A_n(R_1, R_2, \cdots, R_m; T_1, T_2, \cdots, T_m). \quad (23)$$

Equation (20) can then be rewritten

$$T = T(R; R_1, R_2, \cdots, R_m; T_1, T_2, \cdots, T_m), \quad (24)$$

where $T(R_i)$ is the value of T calculated by substituting $R = R_i$ into equation (20). The sensitivity coefficients $\frac{\partial T}{\partial R_i}$ and $\frac{\partial T}{\partial T_i}$ are obtained by differentiating equations (22) and (24) with respect to each calibration point for R_i and T_i, respectively. The results are given as follows:

$$\frac{\partial T}{\partial R_i} = \sum_{j=1}^{n}\left[\mathbf{BH}^{-1}\right]_{i,j}\frac{\partial T}{\partial A_j} \quad (25)$$

and

$$\frac{\partial T}{\partial T_i} = \sum_{j=1}^{n}\left[\mathbf{CH}^{-1}\right]_{i,j}\frac{\partial T}{\partial A_j}. \quad (26)$$

Here, $[\mathbf{M}]_{i,j}$ denotes the (i, j)th element of matrix $[\mathbf{M}]$. Matrix $\mathbf{B}$ has elements

$$\mathbf{B}_{i,j} = -\left.\left(\frac{\partial T}{\partial R}\frac{\partial T}{\partial A_j}\right)\right|_{R=R_i} + \left[T_i - T(R_i)\right]\left.\frac{\partial^2 T}{\partial R \partial A_j}\right|_{R=R_i}$$
$$(i = 1, \cdots, m; j = 1, \cdots, n); \quad (27)$$

likewise, $\boldsymbol{C}$ has elements

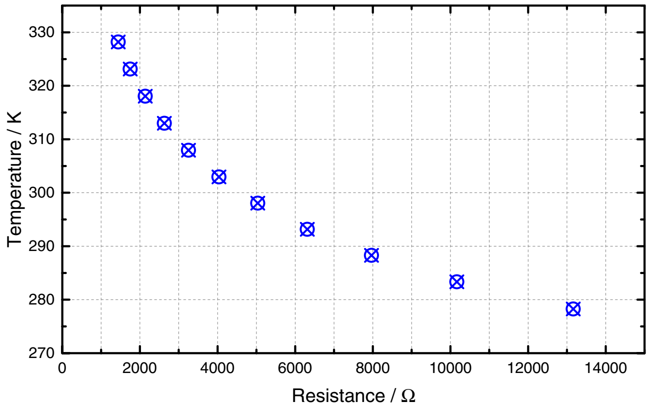

Figure 1. Measured resistance–temperature data for the NTC thermistor over temperature range 278.15–328.15 K.

$$\mathbf{C}_{i,j} = \left.\frac{\partial T}{\partial A_j}\right|_{R=R_i} \quad (i = 1, \cdots, m; j = 1, \cdots, n), \quad (28)$$

and $\mathbf{H}$ has elements

$$\mathbf{H}_{i,j} = \sum_{k=1}^{m}\left\{\left.\left(\frac{\partial T}{\partial A_i}\frac{\partial T}{\partial A_j}\right)\right|_{R=R_k} - \left[T_k - T(R_k)\left.\frac{\partial^2 T}{\partial A_i \partial A_j}\right|_{R=R_k}\right]\right\}$$
$$(i = 1, \cdots, n; j = 1, \cdots, n). \quad (29)$$

The above proposed methodology for obtaining the derivation of sensitivity coefficients can be used for any un-weighted least-squares fitting equations; for a weighted least-squares fitting, equations (21), (22) and (25)–(29) should be slightly modified.

The propagation of uncertainty for temperature T is then calculated from

$$u_T^2 = \sum_{i=1}^{m}\left(\frac{\partial T}{\partial R_i}\right)^2 u_{R_i}^2 + \sum_{i=1}^{m}\left(\frac{\partial T}{\partial T_i}\right)^2 u_{T_i}^2 + \left(\frac{\partial T}{\partial R}\right)^2 u_R^2. \quad (30)$$

2.3. Thermistor calibration system

For this study, we prepared a MF501 NTC thermistor which has a nominal resistance at 25 °C (R_{25}) of about 5 kΩ; its

Table 2. Uncertainty calculation for the calibration points. Note that all the uncertainties are expressed as standard uncertainties ($k = 1$).

Uncertainties contributions		Type	T/K										
			278.2574	283.3417	288.2827	293.1597	298.0455	302.9663	307.9471	312.9821	318.0535	323.1317	328.1941
Uncertainties associated with temperature/mK													
u_1	Reference SPRT calibration	B	2.03	2.06	2.09	2.12	2.15	2.18	2.21	2.24	2.27	2.30	2.33
u_2	Reference SPRT short-term stability	B	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
u_3	Bath temperature non-uniformity	B	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
u_4	Bath temperature stability	B	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40
u_5	Bath temperature drift	B	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
u_6	Reference SPRT readout	B	2.56	2.62	2.68	2.73	2.79	2.85	2.91	2.97	3.03	3.08	3.14
u_7	Self-heating of thermistor	B	0.66	0.51	0.40	0.32	0.25	0.20	0.16	0.13	0.11	0.09	0.07
s_1	Measurement noise of reference SPRT	A	0.64	0.66	0.67	0.68	0.70	0.71	0.73	0.74	0.76	0.77	0.79
u_{Tc}	Combined standard uncertainty of temperature		3.70	3.74	3.79	3.83	3.89	3.95	4.01	4.07	4.13	4.19	4.25
Uncertainties associated with resistance/Ω													
u_8	Calibration thermistor readout	B	0.33	0.25	0.20	0.16	0.13	0.10	0.08	0.07	0.05	0.04	0.04
s_2	Measurement noise of calibration thermistor	A	0.08	0.06	0.05	0.04	0.03	0.03	0.02	0.02	0.01	0.01	0.01
u_{Rc}	Combined standard uncertainty resistance		0.34	0.26	0.21	0.16	0.13	0.10	0.08	0.07	0.05	0.04	0.04

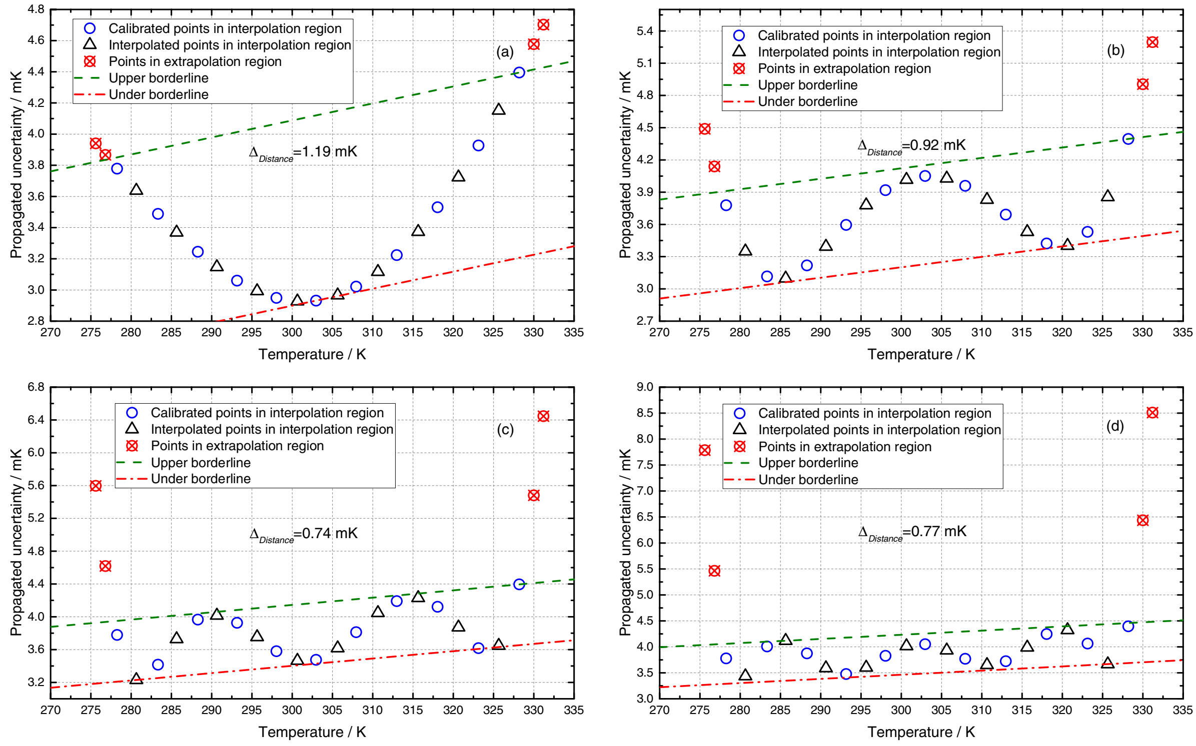

Figure 2. Propagated uncertainties of various thermistor calibration equations obtained using the Lagrange interpolation method. The calibration points used for each panel are listed in table 1. Panel (a): $n = 2$ (Basic equation); Panel (b): $n = 3$ (Hoge-1 equation); Panel (c): $n = 4$ (Hoge-2 equation); Panel (d): $n = 5$ (Hoge-3 equation). Δ_{Distance} in each panel represents the bandwidth between the upper limit and the lower limit.

nominal beta value is 4100 K and its dissipation constant about 2.0–3.0 mW · K^{-1}. The determination of the resistance–temperature characteristics of this thermistor was based on a calibration using the comparison method. The thermistor was inserted into a hermetically sealed glass tube to mitigate secondary effects from placing the thermistor directly into the electric liquid medium (water). A high-precision temperature-controlled water bath (Hart Scientific, Model 7012) was employed to maintain a uniform calibration temperature. The Hart Scientific 7012 bath was calibrated in advance at Jilin Institute of Metrology. Its specifications in regard to stability and uniformity are ±0.8 mK and ±2 mK, respectively, at a water temperature of 298.15 K ($k = 2$). High-precision determination of the thermistor resistance was performed using a Fluke Super-QAD Precision Temperature Scanner Model 1586A (Fluke 1586A) configured with an external DAQ-STAQ multiplexer. The Fluke 1586A was calibrated with standard resistors in the Jilin Institute of Metrology as well, and a relative uncertainty of about 0.005% ($k = 2$) was achieved. Temperature sensing was performed using an SPRT (Model 5628, Hart Scientific) calibrated at Liaoning Institute of Measurement with a standard uncertainty estimated at about 4 mK ($k = 2$) at 273.15 K. Typical measurement currents for the thermistor and the SPRT were 10 μA and 1 mA respectively. The glass tubes used in our experiment were 400 mm in length and 7 mm in diameter. Both glass tubes, with the thermistor and the SPRT, were inserted to the same depth (about 300 mm below the top lips of water) in the bath liquid. Eleven calibration points, uniformly spread at 5 K intervals across the calibration temperature range of 278.15–328.15 K, were preselected.

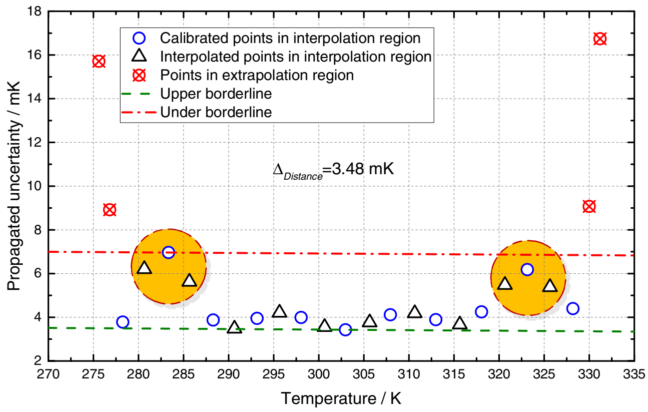

Figure 3. Propagated uncertainties from equation (2) with $n = 6$ obtained using the Lagrange interpolation method. All symbols are as described in figure 2.

3. Results and discussion

3.1. Calibration results

The measured resistance–temperature data for the MF501 NTC thermistor are listed in table 1. Note that T is the average SPRT temperature, and R is the average thermistor resistance, after fully stabilizing the water bath for over half an hour. For

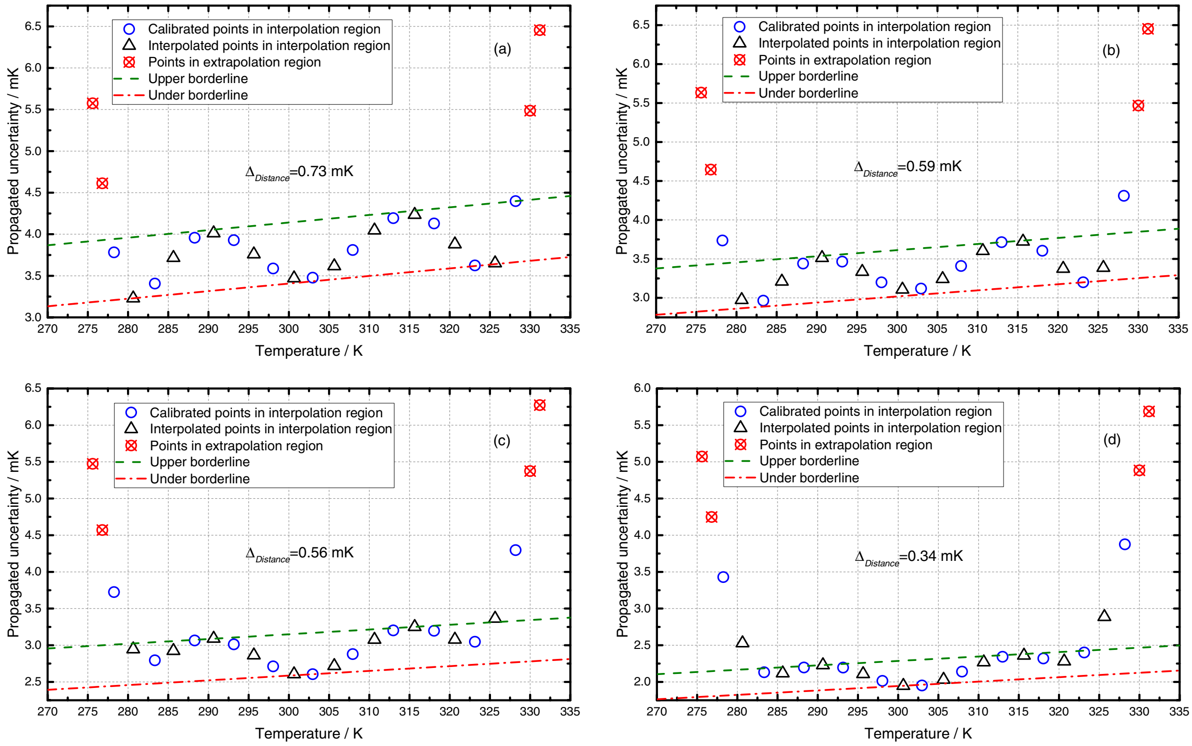

Figure 4. Propagated uncertainties from the Hoge-2 equations with various numbers of calibration points obtained using the least-squares fitting method. Panel (a): $m = 4$ points; Panel (b): $m = 5$ points; Panel (c): $m = 6$ points; Panel (d): $m = 11$ points. The calibration points used for each panel are listed in table 1. Δ_{Distance} in each panel refers to the bandwidth between the upper limit and the lower limit.

the Lagrange interpolation, equation (2) was fitted to 2, 3, 4, 5, and 6 calibration points; for least-squares fitting, equation (5) was fitted to 4, 5, 6, and 11 calibration points. These points are also listed in table 1 for both fitting methods. The measured *R*–*T* data for this thermistor are also plotted in figure 1 for range 278.15–328.15 K at 5.0 K intervals.

The temperature uncertainty and resistance uncertainty for each of the calibration points are calculated based on the manuals provided by the Fluke [19–21]. Note that the measurement noise is the uncertainty caused by the noise or instability of the measurement readings. Each calibration point is based on the average or mean of about 200 readings. Table 2 summarizes the uncertainty calculation for the calibration points.

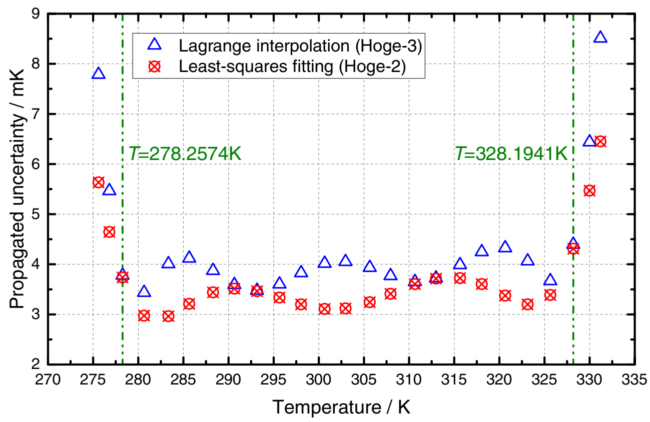

Figure 5. Comparison of the propagated uncertainties from the Hoge-3 and Hoge-2 equations obtained using the Lagrange interpolation and least-squares fitting, with the same five calibration points.

3.2. Propagated uncertainty of Lagrange interpolation

Based on equation (19) and the uncertainties for all calibration points calculated as table 2, the propagated uncertainties corresponding to each of the various thermistor calibration equations (indexed by $n = 2$, 3, 4, 5, and 6 of equation (2)) obtained using the Lagrange interpolation method are shown in figures 2 and 3. Here, we assume that there is a 0.005% ($k = 2$) relative uncertainty in each of the thermistor resistance readouts and one fourth of the uncertainty of the thermistor readout for the noise of the calibration thermistor at the un-calibrated points. To investigate the characteristics associated with the propagation of uncertainty in the extrapolation region, the temperature ranges were extended from about 275 K to 332 K.

There are several interesting features in figures 2 and 3. First, when the number of calibration points is less than four, the propagated uncertainty clearly diminishes as the number of calibration points increases, but only gradually (see Panel (a)–(c) in figure 2). However, the uncertainty becomes slightly larger when calculated using five calibration points with equation (6). Moreover, the bandwidth for equation (2) with $n = 6$ is estimated at almost five times larger than from the Hoge-3 equation (see Panel (d) in figures 2, and 3). According to figure 3, uncertainties at several temperature points within the shaded area were much larger in comparison to the other temperature points. This behavior is known as Runge's phenomenon for high-order interpolations, and this problem may be effectively solved by choosing calibration points corresponding to the Chebyshev nodes [22]. The values of Δ_{Distance}

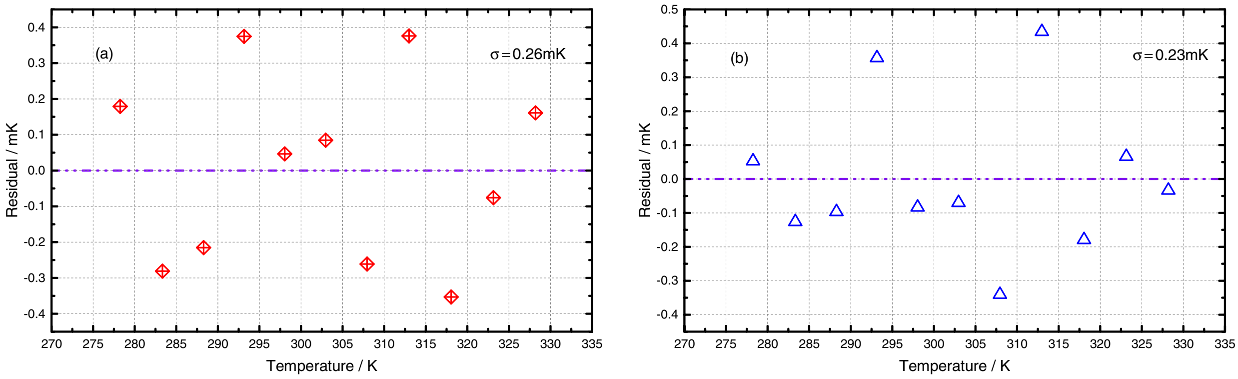

Figure 6. Interpolation residuals from the Hoge-2 and Hoge-3 calibration equations using the least-squares fitting method with 11 calibration points. Panel (a): Hoge-2; Panel (b): Hoge-3.

for the five calibration equations obtained by Lagrange interpolation were found to be 1.19 mK, 0.92 mK, 0.74 mK, 0.77 mK, and 3.48 mK respectively. Second, in the interpolation region, uncertainties at the interpolated points are no greater than those at the calibrated points. The propagated uncertainty is flat in the interpolation region. However, the propagated uncertainty increases very quickly in the extrapolation region. Third, the higher the order of the calibration equation, the faster is the rise in propagated uncertainty in the extrapolation region. Therefore, the propagated uncertainties for the calibration equations determined by Lagrange interpolation are useful only in the interpolation region. Great care is thus required in dealing with calibration points, which need to be chosen appropriately when calculating the propagation of uncertainty based on the Lagrange interpolation method.

3.3. Propagated uncertainty of least-squares fitting

The propagated uncertainties of the Hoge-2 equation equation (5) with numbers of different calibration points using the least-squares fitting method are plotted in figure 4. As before, the relative uncertainty of the thermistor resistance readout is assumed to be 0.005% ($k = 2$), and the noise associated with the calibration thermistor is one-fourth of the uncertainty of the thermistor readout at the uncalibrated points. For figure 4(a), the number of coefficients equals the number of calibration points—this can be treated as an exact fitting obtained by Lagrange interpolation.

The most significant feature in figure 4 is that the propagated uncertainty diminishes rapidly, compared with the Lagrange interpolation, as the number of calibration points increases. The $\Delta_{\mathrm{Distance}}$ value decreases from 0.73 mK for the exact fitting to 0.34 mK for the least-squares fitting with 11 calibration points. In contrast to the Lagrange interpolation, even with only one extra calibration point, the uncertainty can be greatly reduced. Another obvious characteristic is that, unlike the Lagrange interpolation, the propagated uncertainty increases quite quickly towards the end of the interpolation region. Also, the more calibration points used in determining the coefficients of the thermistor calibration equation, the faster the uncertainty increases at the extremes of the calibration range. However, the propagated uncertainty in the extrapolation region seems much less sensitive to the number of calibration points used (see Panel (a) and (d) in figure 4). Similarly to the Lagrange interpolation case, the propagated uncertainty rises quite rapidly in the extrapolation region. Therefore, when the least-squares fitting is used for calibration, and to achieve small propagated uncertainties in a desired measurement range, the calibration range must be slightly larger than the desired measurement range. Moreover, the number of calibration points must be suitably chosen.

To compare the propagated uncertainty resulting from a fitting with the number of coefficients equal to the number of calibration points and a redundant fitting, figure 5 plots the propagation of uncertainties for the Hoge-3 and Hoge-2 equations with the same five calibration points (table 1) using Lagrange interpolation and least-squares fitting, respectively. The result demonstrates that the propagated uncertainty for the Hoge-2 equation obtained by least-squares fitting is much smaller than that for the Hoge-3 equation obtained by Lagrange interpolation, except for those uncertainties at the five calibration points. This result indicates that fitting using least squares is much more robust than that using Lagrange interpolation.

3.4. Interpolation residual

The interpolation residual is defined as the temperature calculated from the calibration equation minus the measured temperature. The propagated uncertainties for the NTC thermistor plotted in figures 2–5 describe only how the calculated temperature is affected by the temperature and resistance uncertainties at the calibration points. However, no account was taken of the differences between the thermistor calibration equation and the true characteristics of the NTC thermistor. To calculate the total uncertainty, the interpolation residual should be considered [9]—especially in the low-order calibration equations, whose interpolation errors can be a major source of uncertainty. One may wonder why the Hoge-2 equation was used instead of Hoge-3 to illustrate the propagated uncertainties with respect to difference in number of calibration points. From our previous study and other research [1, 2], we found that the Hoge-3 equation works only slightly better than Hoge-2; figure 6 shows a slight difference in their interpolation residuals determined by least-squares fitting with the same 11 calibration points. However, the Hoge-2 equation only requires four coefficients to be determined when establishing the resistance–temperature relationship. In addition, based on figure 6, the interpolation errors for both the Hoge-2 equation and Hoge-3 equation can be regarded as negligible in comparison to their propagated uncertainties.

4. Conclusions

We investigated the propagation of uncertainties for several common calibration equations for the NTC thermistor obtained using the Lagrange interpolation and least-squares fitting methods. For this work, a NTC thermistor with a nominal resistance value of about 5 kΩ at a temperature of 298.15 K was calibrated in a precision water bath over a temperature range of 278.15–328.15 K using a comparison method.

Our results show that the propagation of uncertainty is flat in the interpolation region but rises very quickly in the extrapolation region. Within the interpolation region, the propagated uncertainty of the temperature interpolated between calibration points is generally smaller than the uncertainty associated with the calibration points. Therefore, the propagated uncertainty for the calibration equations may only be useful within the calibration range. The results also indicate that the propagation of uncertainty for the calibration equation depends on the number of calibration points. For the Lagrange interpolation, the propagated uncertainty does diminish with increasing number of calibration points, but only gradually. Great care is required in dealing with the choice of calibration points when propagated uncertainty is calculated based on the Lagrange interpolation method. However, for least-squares fitting, the propagated uncertainty decreases significantly as the number of calibration points increases. To achieve smaller propagated uncertainty in a desired measurement range, we recommend the least-squares fitting method with a sufficient number of calibration points and a calibration range slightly larger than the desired measurement range.

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