Research Article | Vol. 25, No. 22 | 30 Oct 2017 | OPTICS EXPRESS 26973

Optics EXPRESS

Three-dimensional polarization ray tracing calculus for partially polarized light

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Abstract: Calculating the evolution of polarization for all polarization states of light in optical systems, in global coordinates, is an important, yet challenging task. This calculation exists for completely polarized light, but has not yet been developed for partially polarized light. A 3 × 3 coherency matrix for partially polarized light, in global coordinates, is presented to calculate the transformation of its polarization as it passes through an optical system. This matrix is a three-dimensional generalization of the coherency matrix. A new coherency matrix calculus method in three dimensions is suggested and validated for two cases. A double Gauss optical lens is introduced to compare this method's performance with two-dimensional calculus.

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OCIS codes: (080.2720) Mathematical methods (general); (110.5405) Polarimetric imaging; (120.5410) Polarimetry; (260.5430) Polarization.

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#302827 https://doi.org/10.1364/OE.25.026973
Journal © 2017 Received 21 Jul 2017; revised 3 Oct 2017; accepted 5 Oct 2017; published 20 Oct 2017

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1. Introduction

Quantifying the evolution of polarization for light passing through optical systems [1–4] is important, especially in the case of partially polarized light passing through polarimetric cameras [5,6] that are used for measuring target polarization. Such analysis allows not only to determine the polarization characteristics of an optical system through which light passes [7–9], but also to determine the polarization of light itself [10–13], which may be affected as a result of light's interaction with polarimetric cameras [14,15].

The evolution of the polarization state can be calculated using the polarization ray tracing technique [16], and many methods for this technique have been developed over the years [17]. Methods that are based on the 2 × 2 Jones matrix and 4 × 4 Mueller matrix have been extensively used for polarization ray tracing [18,19]. The corresponding vectors for representing polarization are the Jones vector (which captures only completely polarized light, but captures the overall phase information) or the coherency matrix, and the Stokes vector (which capture both polarized and partially polarized light, but does not capture the overall phase information). The above methods are all defined for two-dimensional coordinate systems, which means they are only appropriate when the polarization paraxial approximation holds, and ignore the coordinate difference of light rays in different propagation directions.

To use the polarization ray tracing approach on light rays in different propagation directions, Chipman et al. and Yun et al. [20,21] have developed a method that uses a 3 × 1 polarization vector and a 3 × 3 polarization matrix, which are three-dimensional generalizations of the Jones vector and Jones matrix. However, this method can only describe the evolution of completely polarized light, as the Jones vector cannot describe the polarization state of unpolarized and partially polarized light. Jose J. Gil et al. [22,23] proposed the 3D coherency matrices to describe the degree of polarization in three dimensions, which consist of generalized Stokes parameters, but they didn't focus on the evolution of polarization in optical systems. Because in many situations light is only partially polarized, it becomes more important to calculate the polarization evolution of such partially polarized light. In particular, a method is urgently needed to perform calculations for polarimetric cameras; the results of these calculations can then be used for calibrating the polarization effects of polarimetric cameras [11–13], which would be advantageous for improving their measurement accuracy.

In this article, we describe the development of the three-dimensional polarization ray tracing calculus for partially polarized light, using a new 3 × 3 coherency matrix. Unlike the previous 3D coherency matrices, the 3 × 3 coherency matrix is a three-dimensional generalization of the classical coherency matrix, and is used to describe partially and fully polarized states in three-dimensional global coordinates. The physical associations between this novel coherency matrix for global coordinates and the currently used coherency matrix are described. The calculus of three-dimensional polarization ray-tracing for partially polarized light is derived in detail. The method is validated on two special cases: 1) global coordinates are aligned with the local coordinates of incident and exiting light, 2) linear polarized light is refracted by a dielectric medium.

This paper is organized as follows. In Section 2, we describe the previous polarization calculus and the associated vectors and matrices available for describing the polarization of light. In Section 3, the novel 3 × 3 coherency matrix in global coordinates is proposed. In Section 4, the calculus of three-dimensional polarization ray tracing for partially polarized light is derived using the proposed 3 × 3 coherency matrix. In Section 5, the method is demonstrated and validated on two special cases and one example. In Section 6, we summarize and conclude the paper.

2. Previous polarization calculus

In this section, we summarize the previous polarization calculus and review the definition of the currently used coherency matrix.

2.1 Summary

Table 1 lists the previous polarization calculus and the associated vectors and matrices available for describing the polarization of light.

Table 1. The previous polarization calculus.

Calculus and properties	Polarization vector	Polarization matrix	Polarization evolution equation
Jones calculus Amplitude calculus Only polarized light Local coordinates	Jones vector $\mathbf{E}=\begin{bmatrix}E_x\\E_y\end{bmatrix}$	Jones matrix $\mathbf{T}=\begin{bmatrix}T_{11} & T_{12}\\T_{21} & T_{22}\end{bmatrix}$	$\mathbf{E}_{\mathbf{(out)}}=\mathbf{T}\times\mathbf{E}_{\mathbf{(in)}}$
Coherency matrix calculus Intensity calculus All polarization state Local coordinates	Coherency matrix $\mathbf{J}_\mathbf{P}=\begin{bmatrix}J_{xx} & J_{xy}\\J_{yx} & J_{yy}\end{bmatrix}$	Jones matrix $\mathbf{J}_\mathbf{P}=\begin{bmatrix}J_{xx} & J_{xy}\\J_{yx} & J_{yy}\end{bmatrix}$	$\mathbf{J}_{\mathbf{P(out)}}=\mathbf{T}\times\mathbf{J}_{\mathbf{P(in)}}\times\mathbf{T}^\dagger$
Mueller calculus Intensity calculus All polarization state Local coordinates	Stokes vector $\mathbf{S}=\begin{bmatrix}S_0\\S_1\\S_2\\S_3\end{bmatrix}$	Mueller matrix $\mathbf{M}=\begin{bmatrix}m_{00} & m_{01} & m_{02} & m_{03}\\m_{10} & m_{11} & m_{12} & m_{13}\\m_{20} & m_{21} & m_{22} & m_{23}\\m_{30} & m_{31} & m_{32} & m_{33}\end{bmatrix}$	$\mathbf{S}_{\mathbf{(out)}}=\mathbf{M}\times\mathbf{S}_{\mathbf{(in)}}$
Polarization ray tracing calculus Amplitude calculus Only polarized light Global coordinates	Polarization vector $\mathbf{E_3}=\begin{bmatrix}E_x\\E_y\\E_z\end{bmatrix}$	Polarization ray tracing matrix $\mathbf{P_3}=\begin{bmatrix}p_{11} & p_{12} & p_{13}\\p_{21} & p_{22} & p_{23}\\p_{31} & p_{32} & p_{33}\end{bmatrix}$	$\mathbf{E}_{3\mathbf{(out)}}=\mathbf{P_3}\times\mathbf{E}_{3\mathbf{(in)}}$

We can see from Table 1 that the existing polarization calculi utilize the Jones matrix or the Mueller matrix. These are well known and are described in detail elsewhere. The polarization vector associated with the Mueller matrix is the Stokes vector, and it can represent both polarized and partially polarized light. There are two polarization vectors associated with the Jones matrix, i.e., the Jones vector and the coherency matrix. Because the Jones vector can only represent polarized light, the coherency matrix can be a suitable substitution for the Jones matrix.

Local coordinates indicate that the polarization calculus is performed in two-dimensional coordinates, while global coordinates indicate that the calculus is performed in three-dimensional coordinates. Three-dimensional calculus is more objective, because two-dimensional calculus assumes polarization paraxial approximation. One three-dimensional method was suggested by Yun et al. [20,21]; this method is called the polarization ray tracing calculus. It is a three-dimensional generalization of the Jones calculus, including the generalization of the Jones vector and Jones matrix. It can describe the polarization evolution of light as it passes through an optical system in three-dimensional global coordinates. However, the method cannot describe the polarization evolution for partially polarized light, because the generalization of the Jones vector can only describe polarized light.

2.2 Coherency matrix

The coherency matrix is described in detail by Born and Wolf in their books [24,25]. Here we just review its definition and properties.

Consider a quasi-monochromatic light wave that propagates in the positive z direction. Let E_x and E_y represent the components of the electric field vector

$$\mathbf{E} = A \cdot e^{i(kz-wt)} \tag{1}$$

in two mutually orthogonal directions, which are perpendicular to the direction of propagation. The coherency matrix $\mathbf{J_P}$ of a quasi-monochromatic light wave is defined by

$$\begin{aligned}\mathbf{J_P} &= \left\langle \mathbf{E} \times \mathbf{E}^{\dagger} \right\rangle \\ &= \left\langle \begin{bmatrix} E_x \\ E_y \end{bmatrix} \times \left[E_x^*, E_y^* \right] \right\rangle \\ &= \begin{bmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{bmatrix}.\end{aligned} \tag{2}$$

Here, the operator $\langle \ \rangle$ denotes averaging over some short window of time, the operator $\times$ denotes the matrix multiplication, the superscript $\dagger$ is the Hermitian conjugate, and the superscript * denotes conjugate. The diagonal elements of $\mathbf{J_P}$, J_{xx} and J_{yy}, are real and represent the intensities of the components in the x and y directions. Hence, the trace $Tr(\mathbf{J_P})$ of the matrix is equal to the total intensity of the light wave,

$$Tr(\mathrm{J_P}) = \left\langle E_x E_x^* \right\rangle + \left\langle E_y E_y^* \right\rangle = J_{xx} + J_{yy}. \tag{3}$$

The nondiagonal elements are in general complex, but they are conjugates of each other.

The degree of polarization P (hereafter denoted as DoP) of the wave is given by

$$P = \sqrt{1 - \frac{4\left|\mathrm{J_P}\right|}{\left(J_{xx} + J_{yy}\right)^2}}, \quad 0 \le P \le 1. \tag{4}$$

Here, the operator $\left|\mathbf{J_P}\right|$ denotes the associated determinant of the coherency matrix $\mathbf{J_P}$.

For $P = 1$, the wave is completely polarized. For a general polarization state (i.e., partially elliptically polarized light), the orientation θ (called Orient in this paper) of polarization and the ellipticity angle ε are given by

$$\theta = \frac{1}{2}\arctan\left(\frac{J_{xy} + J_{yx}}{J_{xx} - J_{yy}}\right), \quad \varepsilon = \frac{1}{2}\arcsin\left(\frac{-j\left(J_{xy} - J_{yx}\right)}{P\left(J_{xx} + J_{yy}\right)}\right). \tag{5}$$

In two-dimensional coordinates, the coherency matrix [26] of partially elliptically polarized light with intensity I is

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$$\mathbf{J_P} = \frac{I}{2}\begin{bmatrix} 1+P\cos 2\theta\cos 2\varepsilon & P(\sin 2\theta\cos 2\varepsilon - j\sin 2\varepsilon) \\ P(\sin 2\theta\cos 2\varepsilon + j\sin 2\varepsilon) & 1-P\cos 2\theta\cos 2\varepsilon \end{bmatrix}. \tag{6}$$

When the ellipticity angle is $\varepsilon = 0°$, partially elliptically polarized light becomes partially linearly polarized light.

3. New 3 × 3 coherency matrix in global coordinates

In this section, we present a new 3 × 3 coherency matrix in three-dimensional global coordinates.

3.1 Coherency matrix in the eigen-vibration coordinates

From the above descriptions, the coherency matrix in two-dimensional coordinates is defined in the plane perpendicular to the direction of propagation, which we here call the "eigen-vibration plane" of the propagating light. We can build a right-handed three-dimensional coordinate system using the eigen-vibration plane and the propagation vector; in what follows, we call these coordinates the "eigen-vibration coordinates" of the propagating light. Each light ray has independent eigen-vibration coordinates, O-X'Y'Z', as the propagation vectors of light are different in an optical system. The coordinates are three-dimensional local coordinates. A schematic is shown in Fig. 1.

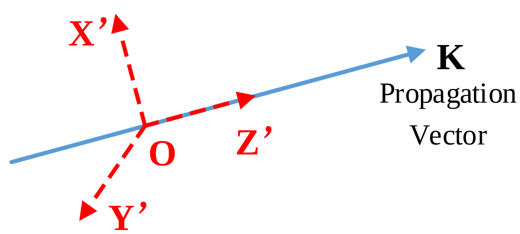

Fig. 1. The eigen-vibration coordinates of the propagating light.

In the eigen-vibration coordinates, a three-dimensional generalization of the coherency matrix is easy to derive. Let $E_{x'}$, $E_{y'}$, and $E_{z'}$ represent the three components of the electric field vector $\mathbf{E_L}$. The $E_{z'}$ component is always zero, because the electric field vector only oscillates in the eigen-vibration plane. The coherency matrix $\mathbf{J_{P3L}}$ in the ray's eigen-vibration coordinates can be rewritten as

$$\begin{aligned} \mathbf{J_{P3L}} &= \left\langle \mathbf{E_L} \times \mathbf{E_L}^{\dagger} \right\rangle \\ &= \left\langle \begin{bmatrix} E_{x'} \\ E_{y'} \\ 0 \end{bmatrix} \times \left[E_{x'}^{*}, E_{y'}^{*}, 0 \right] \right\rangle \\ &= \begin{bmatrix} \mathbf{J_P} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{7}$$

The trace $Tr(\mathbf{J_{P3L}})$ of the matrix is also equal to the total intensity of the propagating light. Because the associated determinant $|\mathbf{J_{P3L}}|$ is always zero, the degree of polarization P in eigen-vibration coordinates should be rewritten as

$$P = \sqrt{1 - \frac{4(J_{xx}J_{yy} - J_{xy}J_{yx})}{\left(J_{xx} + J_{yy}\right)^2}}. \tag{8}$$

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The other polarization parameters are given in Eq. (5).

3.2 Coherency matrix in three-dimensional global coordinates

In an optical system, different light rays will propagate in different directions. Each light ray has its own coherency matrix $\mathbf{J}_{P3L}$ in its own eigen-vibration coordinates. It then becomes necessary to express all coherency matrices $\mathbf{J}_{P3}$ in the same three-dimensional global coordinate system.

To accomplish this process, we first transform local coordinates into global coordinates. We assume that the propagation vector of light is $\mathbf{K} = [K_x, K_y, K_z]^T$ in the global coordinates O-XYZ. The eigen-vibration coordinates O-X'Y'Z' are constructed relative to the light ray's propagation vector. A schematic is shown in Fig. 2.

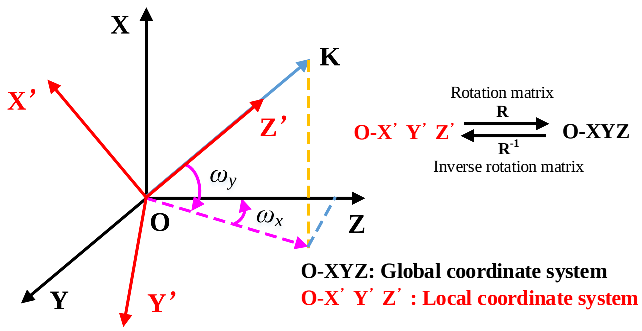

Fig. 2. Global coordinates O-XYZ and the eigen-vibration coordinates O-X'Y'Z'.

The transformation from local coordinates into global coordinates is as follows: 1) the eigen-vibration coordinates rotate ω_y around the Y' axis and then 2) rotate ω_x around the X' axis. The rotation angles ω_y and ω_x are given by

$$\omega_y = -(90 - \arccos(\frac{K_x}{\sqrt{K_x^2 + K_y^2 + K_z^2}})), \quad \omega_x = \arctan(\frac{K_y}{K_z}). \tag{9}$$

The rotation matrix $\mathbf{R}$ is

$$\mathbf{R} = \mathbf{R}_x \times \mathbf{R}_y, \tag{10}$$

where

$$\mathbf{R}_y = \begin{bmatrix} \cos\omega_y & 0 & -\sin\omega_y \\ 0 & 1 & 0 \\ \sin\omega_y & 0 & \cos\omega_y \end{bmatrix}, \quad \mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_x & \sin\omega_x \\ -\sin\omega_x & 0 & \cos\omega_x \end{bmatrix}. \tag{11}$$

By using the rotation matrix $\mathbf{R}$, we can transform the coherency matrix from eigen-vibration coordinates into global coordinates. The electric field vector $\mathbf{E}_G$ in global coordinates is

$$\mathbf{E}_G = \mathbf{R} \times \mathbf{E}_L. \tag{12}$$

The coherency matrix $\mathbf{J}_{P3}$ in the three-dimensional global coordinate system is

$$\begin{aligned} \mathbf{J}_{P3} &= \left\langle \mathbf{E}_G \times \mathbf{E}_G{}^{\dagger} \right\rangle \\ &= \left\langle \mathbf{R} \times \mathbf{E}_L \times (\mathbf{R} \times \mathbf{E}_L^{*})^T \right\rangle \\ &= \mathbf{R} \times \mathbf{J}_{P3L} \times \mathbf{R}^T. \end{aligned} \tag{13}$$

For all polarization states, the coherency matrix $\mathbf{J}_{\mathbf{P3}}$ can be obtained from the associated rotation matrix $\mathbf{R}$ and coherency matrix $\mathbf{J}_{\mathbf{P3L}}$ in the corresponding eigen-vibration coordinates. The correctness of this statement can be demonstrated by decomposing polarization states into completely polarized light, completely unpolarized light (natural light), and partially polarized light.

Firstly, we take a beam of completely polarized light as an example. In the eigen-vibration coordinates, its Jones vector is $\mathbf{E}_{\mathbf{3L}}$ and its coherency matrix is $\mathbf{J}_{\mathbf{CP3L}}$. In the global coordinates, its Jones vector $\mathbf{E}_{\mathbf{3}}$ can be calculated as $\mathbf{E}_3 = \mathbf{R} \times \mathbf{E}_{3L}$. Its coherency matrix $\mathbf{J}_{\mathbf{CP3}}$ in three-dimensional global coordinates is calculated as

$$\begin{aligned}\mathbf{J}_{\mathbf{CP3}} &= \left\langle \mathbf{E}_3 \times \mathbf{E}_3^{\dagger} \right\rangle \\ &= \left\langle \mathbf{E}_3 \times (\mathbf{E}_3^*)^T \right\rangle \\ &= \left\langle \mathbf{R} \times \mathbf{E}_{3L} \times (\mathbf{R} \times \mathbf{E}_{3L}^*)^T \right\rangle \\ &= \left\langle \mathbf{R} \times \mathbf{E}_{3L} \times (\mathbf{E}_{3L}^*)^T \times \mathbf{R}^T \right\rangle \\ &= \mathbf{R} \times \mathbf{J}_{\mathbf{CP3L}} \times \mathbf{R}^T.\end{aligned} \tag{14}$$

Secondly, we discuss about a beam of completely unpolarized light (natural light) with intensity I_{up}. The natural light can be decomposed into the addition of two independent linear polarized light [24]. In the two-dimensional coordinates (the eigen-vibration plane), the coherency matrix of the natural light $\mathbf{J}_{\mathbf{UP}}$ can be expressed as

$$\mathbf{J}_{\mathbf{UP}} = \frac{I_{up}}{2}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \frac{I_{up}}{2}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}. \tag{15}$$

In the eigen-vibration coordinates, its coherency matrix $\mathbf{J}_{\mathbf{UP3L}}$ can be expressed as

$$\mathbf{J}_{\mathbf{UP3L}} = \frac{I_{up}}{2}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} + \frac{I_{up}}{2}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{16}$$

In the global coordinates, its coherency matrix $\mathbf{J}_{\mathbf{UP3}}$ is calculated as

$$\mathbf{J}_{\mathbf{UP3}} = \mathbf{R} \times \mathbf{J}_{\mathbf{UP3L}} \times \mathbf{R}^T. \tag{17}$$

Thirdly, we discuss about a beam of partially polarized light. In the eigen-vibration coordinates, its coherency matrix, $\mathbf{J}_{\mathbf{P3L}}(P, \theta, \varepsilon)$, with intensity I, can be decomposed into the addition of two coherency matrixes. One is the coherency matrix of completely polarized light, $\mathbf{J}_{\mathbf{CP3L}}$, with intensity PI. Another is the coherency matrix of completely unpolarized light, $\mathbf{J}_{\mathbf{UP3L}}$, with intensity $(1\text{-}P)I$. They can be described as

$$\begin{aligned}
&\mathbf{J}_{\mathbf{P3L}}=\frac{I}{2}\begin{bmatrix}1+P\cos2\theta\cos2\varepsilon & P(\sin2\theta\cos2\varepsilon-j\sin2\varepsilon) & 0\\ P(\sin2\theta\cos2\varepsilon+j\sin2\varepsilon) & 1-P\cos2\theta\cos2\varepsilon & 0\\ 0 & 0 & 0\end{bmatrix},\\
&\mathbf{J}_{\mathbf{CP3L}}=\frac{PI}{2}\begin{bmatrix}1+\cos2\theta\cos2\varepsilon & \sin2\theta\cos2\varepsilon-j\sin2\varepsilon & 0\\ \sin2\theta\cos2\varepsilon+j\sin2\varepsilon & 1-\cos2\theta\cos2\varepsilon & 0\\ 0 & 0 & 0\end{bmatrix},\\
&\mathbf{J}_{\mathbf{UP3L}}=\frac{(1-P)I}{2}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\\
&\mathbf{J}_{\mathbf{P3L}}=\mathbf{J}_{\mathbf{CP3L}}+\mathbf{J}_{\mathbf{UP3L}}.
\end{aligned}\tag{18}$$

In the global coordinates, its coherency matrix $\mathbf{J}_{\mathbf{P3}}$ is also the addition of the two associated coherency matrixes, which can be described as

$$\mathbf{J}_{\mathbf{P3}}=\mathbf{J}_{\mathbf{CP3}}+\mathbf{J}_{\mathbf{UP3}},\tag{19}$$

where

$$\begin{aligned}\mathbf{J}_{\mathbf{CP3}}&=\mathbf{R}\times\mathbf{J}_{\mathbf{CP3L}}\times\mathbf{R}^T,\\ \mathbf{J}_{\mathbf{UP3}}&=\mathbf{R}\times\mathbf{J}_{\mathbf{UP3L}}\times\mathbf{R}^T.\end{aligned}\tag{20}$$

Hence the coherency matrix $\mathbf{J}_{\mathbf{P3}}$ is obtained as

$$\begin{aligned}\mathbf{J}_{\mathbf{P3}}&=\mathbf{R}\times\mathbf{J}_{\mathbf{CP3L}}\times\mathbf{R}^T+\mathbf{R}\times\mathbf{J}_{\mathbf{UP3L}}\times\mathbf{R}^T\\ &=\mathbf{R}\times(\mathbf{J}_{\mathbf{CP3L}}+\mathbf{J}_{\mathbf{UP3L}})\times\mathbf{R}^T\\ &=\mathbf{R}\times\mathbf{J}_{\mathbf{P3L}}\times\mathbf{R}^T.\end{aligned}\tag{21}$$

Because the polarization parameters (i.e., the degree of polarization P, the orientation of polarization θ, and the ellipticity angle ε) are defined in the eigen-vibration plane, they cannot be calculated in global coordinates. To calculate these, the coherency matrix should be transformed from global coordinates into the system of eigen-vibration coordinates, using the inverse matrix $\mathbf{R}^{-1}$ of the rotation matrix. After expressing the coherency matrix in the system of eigen-vibration coordinates, the polarization parameters can be calculated as in Eqs. (5) and (8).

4. Three-dimensional polarization ray tracing calculus for partially polarized light

In this section, we develop the calculus of three-dimensional polarization ray tracing for partially polarized light.

After obtaining the polarization vector and the associated polarization matrix, the polarization ray tracing calculus can be developed using the associated polarization evolution equation. For partially polarized light, we have decided to use the coherency matrix calculus for polarization ray tracing. In two-dimensional coordinates, the coherency matrix calculus is associated with the coherency matrix and the Jones matrix. In three-dimensional global coordinates, the associated Jones matrix should be generalized to three-dimensional global coordinates. This has been accomplished by Yun et al. [20], and the polarization ray tracing matrix $\mathbf{P}_3$ is the generalization of the Jones matrix in global coordinates.

After obtaining the generalization of the coherency matrix $\mathbf{J}_{\mathbf{P3}}$ and the Jones matrix $\mathbf{P}_3$ in three-dimensional global coordinates, three-dimensional polarization ray tracing calculus for partially polarized light can be developed. The associated polarization evolution equation at the medium surface q is rewritten as

$$\mathbf{J}_{\mathbf{P3(out,\,q)}} = \mathbf{P}_{\mathbf{3(q)}} \times \mathbf{J}_{\mathbf{P3(in,\,q)}} \times \mathbf{P}_{\mathbf{3(q)}}{}^{\dagger}. \tag{22}$$

The polarization ray tracing matrix $\mathbf{P}_{\mathbf{3(q)}}$ represents the polarization matrix of the medium surface q in the three-dimensional global coordinate system [20]. It is given by

$$\mathbf{P}_{\mathbf{3(q)}} = \left[\mathbf{p}_{\mathbf{(out,\,q)}},\, \mathbf{s}_{\mathbf{(out,\,q)}},\, \mathbf{k}_{\mathbf{(out,\,q)}}\right] \times \mathbf{T}_{\mathbf{3(q)}} \times \left[\mathbf{p}_{\mathbf{(in,\,q)}},\, \mathbf{s}_{\mathbf{(in,\,q)}},\, \mathbf{k}_{\mathbf{(in,\,q)}}\right]^{T}, \tag{23}$$

where

$$\begin{gathered}\vec{\mathbf{s}}_{\mathbf{(in,\,q)}} = \frac{\vec{\mathbf{k}}_{\mathbf{(in,\,q)}} \times \vec{\mathbf{k}}_{\mathbf{(out,\,q)}}}{\left|\vec{\mathbf{k}}_{\mathbf{(in,\,q)}} \times \vec{\mathbf{k}}_{\mathbf{(out,\,q)}}\right|},\quad \vec{\mathbf{p}}_{\mathbf{(in,\,q)}} = \vec{\mathbf{k}}_{\mathbf{(in,\,q)}} \times \vec{\mathbf{s}}_{\mathbf{(in,\,q)}}, \\ \vec{\mathbf{s}}_{\mathbf{(out,\,q)}} = \vec{\mathbf{s}}_{\mathbf{(in,\,q)}},\quad \vec{\mathbf{p}}_{\mathbf{(out,\,q)}} = \vec{\mathbf{k}}_{\mathbf{(out,\,q)}} \times \vec{\mathbf{s}}_{\mathbf{(out,\,q)}}.\end{gathered} \tag{24}$$

The 3 × 3 Jones matrix $\mathbf{T}_{\mathbf{3(q)}}$ is the generalization of the Jones matrix $\mathbf{T}_{\mathbf{(q)}}$ in the surface local coordinates $\{\mathbf{s}_{\mathbf{(q)}}, \mathbf{p}_{\mathbf{(q)}}, \mathbf{k}_{\mathbf{(q)}}\}$.

For dielectrics, metals, multilayer coated media and other isotropic media, the matrix is defined by

$$\mathbf{T}_{\mathbf{3(q)}} = \begin{bmatrix} \mathbf{T}_{\mathbf{(q)}} & 0 \\ 0 & 1 \end{bmatrix},\ \mathbf{T}_{\mathbf{(q)}} = \begin{bmatrix} t_{p(q)} & 0 \\ 0 & t_{s(q)} \end{bmatrix} \text{ or } \begin{bmatrix} r_{p(q)} & 0 \\ 0 & r_{s(q)} \end{bmatrix}. \tag{25}$$

The quantities t and r are refraction and reflection, respectively, for the surface q. The coefficients $t_{s(q)}$, $t_{p(q)}$ are s- and p-amplitude transmission coefficients, while $r_{s(q)}$, $r_{p(q)}$ are reflection coefficients. For an uncoated interface between two isotropic media, they can be calculated from the Fresnel equations [27]. For coated interfaces, they can be calculated from multilayer coating calculations.

For the surfaces of gratings, holograms, subwavelength gratings, and other nonisotropic media, the definition of the 3 × 3 Jones matrix $\mathbf{T}_{\mathbf{3(q)}}$ has been provided previously [28,29].

Because a typical optical system has more than one surface, Eq. (23) is used for every surface. The total polarization ray tracing matrix $\mathbf{P}_{\mathbf{3(total)}}$ of an optical system is

$$\mathbf{P}_{\mathbf{3(total)}} = \prod_{q=1}^{Q} \mathbf{P}_{\mathbf{3(q)}}. \tag{26}$$

According to Eq. (22), the coherency matrix $\mathbf{J}_{\mathbf{P3(out)}}$ for the final exiting light is

$$\mathbf{J}_{\mathbf{P3(out)}} = \mathbf{P}_{\mathbf{3(total)}} \times \mathbf{J}_{\mathbf{P3(in)}} \times \mathbf{P}_{\mathbf{3(total)}}{}^{\dagger}, \tag{27}$$

where the coherency matrix $\mathbf{J}_{\mathbf{P3(in)}}$ represents the polarization state of the incident light.

5. Demonstration and application example

In this section, we assess the validity of the proposed method on two special situations, and illustrate the method at work by using a double Gauss optical lens.

5.1 Relationship with the coherency matrix calculus in two-dimensional coordinates

From the discussion in section 2, the calculus in two-dimensional coordinates is based on the polarization paraxial approximation. This implies that in the global coordinate system, all light rays in an optical system look like to propagate along the z axis when their evolutions of polarization are calculated. A schematic is shown in Fig. 3.

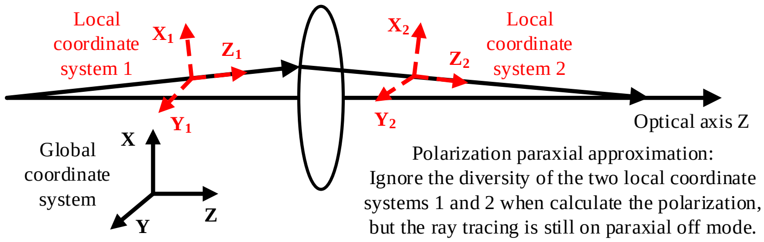

Fig. 3. Polarization paraxial approximation for an optical system.

Let the polarization vectors before and after a medium's surface be $\mathbf{k}_{(\mathbf{in})}$ and $\mathbf{k}_{(\mathbf{out})}$, which can be expressed as

$$\vec{\mathbf{k}}_{(\mathbf{in})} = [0, 0, 1]^T, \ \vec{\mathbf{k}}_{(\mathbf{out})} = [0, 0, 1]^T. \tag{28}$$

Because, in this case, the global coordinates are aligned with local coordinates, the associated rotation matrices $\mathbf{R}_{(\mathbf{in})}$ and $\mathbf{R}_{(\mathbf{out})}$ for the two eigen-vibration coordinates are identity matrices. The coherency matrix $\mathbf{J}_{\mathbf{P3(in)}}$ of the incident light and the polarization ray tracing matrix $\mathbf{P}_3$ in global coordinates can be expressed using the associated coherency matrix $\mathbf{J}_{\mathbf{P(in)}}$ and the Jones matrix $\mathbf{T}$ in two-dimensional coordinates as

$$\mathbf{J}_{\mathbf{P3(in)}} = \begin{bmatrix} \mathbf{J}_{\mathbf{P(in)}} & 0 \\ 0 & 0 \end{bmatrix}, \ \mathbf{P}_3 = \begin{bmatrix} \mathbf{T} & 0 \\ 0 & 1 \end{bmatrix}. \tag{29}$$

The coherency matrix $\mathbf{J}_{\mathbf{P3(out)}}$ for the light exiting from the medium's surface in global coordinates is calculated as

$$\begin{aligned} \mathbf{J}_{\mathbf{P3(out)}} &= \mathbf{P}_3 \times \mathbf{J}_{\mathbf{P3(in)}} \times \mathbf{P}_3^{\dagger} \\ &= \begin{bmatrix} \mathbf{T} \times \mathbf{J}_{\mathbf{P(in)}} \times \mathbf{T}^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{30}$$

The associated coherency matrix $\mathbf{J}_{\mathbf{P(out)}}$ in two-dimensional coordinates is

$$\mathbf{J}_{\mathbf{P(out)}} = \mathbf{T} \times \mathbf{J}_{\mathbf{P(in)}} \times \mathbf{T}^{\dagger}. \tag{31}$$

The coherency matrix $\mathbf{J}_{\mathbf{P3L(out)}}$ in the eigen-vibration coordinate system is also equal to the coherency matrix $\mathbf{J}_{\mathbf{P3(out)}}$ because the associated rotation matrix $\mathbf{R}_{(\mathbf{out})}{}^{-1}$ is the identity matrix. We can see from Eqs. (30) and (31) that the polarization parameters represented by the coherency matrix $\mathbf{J}_{\mathbf{P3(out)}}$ are equivalent to those represented by the coherency matrix $\mathbf{J}_{\mathbf{P(out)}}$ in two-dimensional coordinates.

The above derivations indicate that the coherency matrix calculus in two-dimensional coordinates is just a special situation of the one in global coordinates, which can be taken as a demonstration of the correctness of the proposed method.

5.2 Relationship with the polarization ray tracing calculus

The polarization ray tracing calculus proposed by Yun et al. is used to describe the evolution of polarized light as it passes through an optical system, in three-dimensional global coordinates. The method of calculation proposed in this paper describes the evolution of all polarization states. When they are all used to describe the evolution of polarized light, we can compare the two results to demonstrate the correctness of the proposed method.

Consider a beam of completely linearly polarized light, with the polarization orientation θ and unit intensity (I = 1). In the system of eigen-vibration coordinates, its Jones vector $\mathbf{E}_{\mathbf{3L(in)}}$ and coherency matrix $\mathbf{J}_{\mathbf{CP3L(in)}}$ are expressed as

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$$\mathbf{E}_{\mathbf{3L(in)}} = \left[\cos\theta,\ \sin\theta,\ 0\right]^T,$$
$$\mathbf{J}_{\mathbf{CP3L(in)}} = \frac{1}{2}\begin{bmatrix} 1+\cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & 1-\cos 2\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{32}$$

The propagation vectors before and after a medium's surface are also $\mathbf{k}_{\mathbf{(in)}}$ and $\mathbf{k}_{\mathbf{(out)}}$. The transformation from local coordinates of $\mathbf{k}_{\mathbf{(in)}}$ into global coordinates is represented by the rotation matrix $\mathbf{R}_{\mathbf{(in)}}$ (see Fig. 2). It is given in Eq. (10). The transformation from global coordinates back into local coordinates of $\mathbf{k}_{\mathbf{(out)}}$ is represented by the inverse rotation matrix $\mathbf{R}_{\mathbf{(out)}}{}^{-1}$ (see Fig. 2). It is inverse of $\mathbf{R}$.

According to Eqs. (7)-(27), the Jones vector $\mathbf{E}_{\mathbf{3L(out)}}$ and the coherency matrix $\mathbf{J}_{\mathbf{CP3L(out)}}$ of the exiting light, in their eigen-vibration coordinates, are

$$\mathbf{E}_{\mathbf{3L(out)}} = \mathbf{R}^{-1}_{\mathbf{(out)}} \times \mathbf{P}_{\mathbf{3}} \times \mathbf{R}_{\mathbf{(in)}} \times \mathbf{E}_{\mathbf{3L(in)}},$$
$$\mathbf{J}_{\mathbf{CP3L(out)}} = \mathbf{R}^{-1}_{\mathbf{(out)}} \times [\mathbf{P}_{\mathbf{3}} \times (\mathbf{R}_{\mathbf{(in)}} \times \mathbf{J}_{\mathbf{CP3L(in)}} \times \mathbf{R}^{T}_{\mathbf{(in)}}) \times \mathbf{P}_{\mathbf{3}}{}^{\dagger}] \times (\mathbf{R}^{-1}_{\mathbf{(out)}})^T. \tag{33}$$

The two polarization orientations $\theta_{1(\mathbf{E3})}$ and $\theta_{1(\mathbf{J3})}$ represented by $\mathbf{E}_{\mathbf{3L(out)}}$ and $\mathbf{J}_{\mathbf{CP3L(out)}}$ are

$$\theta_{1(\mathbf{E}_3)} = \arctan\left[\frac{\mathbf{E}_{\mathbf{3L(out)}}(2,1)}{\mathbf{E}_{\mathbf{3L(out)}}(1,1)}\right],$$
$$\theta_{1(\mathbf{J}_3)} = \frac{1}{2}\arctan\left[\frac{\mathbf{J}_{\mathbf{CP3L(out)}}(1,2)+\mathbf{J}_{\mathbf{CP3L(out)}}(2,1)}{\mathbf{J}_{\mathbf{CP3L(out)}}(1,1)-\mathbf{J}_{\mathbf{CP3L(out)}}(2,2)}\right], \tag{34}$$

where the tuple (row, column) represents an element in the associated vector or matrix.

We can assign random values to the parameters, i.e., to the polarization orientation θ of the incident light, the two polarization vectors $\mathbf{k}_{\mathbf{(in)}}$, $\mathbf{k}_{\mathbf{(out)}}$, and the two amplitude transmission coefficients t_s, t_p. The calculation results for the two quantities in Eq. (34) are the same for the two methods, validating the proposed method. A numerical example is listed in Table 2.

Table 2. A numerical example of the two three-dimensional calculi (θ = 45°).

	$\mathbf{E}_{\mathbf{3L(in)}}$ $\mathbf{J}_{\mathbf{CP3L(in)}}$	$\mathbf{K}_{\mathbf{(in)}}$	$\mathbf{K}_{\mathbf{(out)}}$	$\mathbf{P}_{\mathbf{3}}$	$\mathbf{E}_{\mathbf{3L(out)}}$ $\mathbf{J}_{\mathbf{CP3L(out)}}$	$\theta_{1(\mathbf{E3})}$ $\theta_{1(\mathbf{J3})}$
Jones vector	$\frac{\sqrt{2}}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}$	$\begin{bmatrix}0.0485\\0.1217\\0.9914\end{bmatrix}$	$\begin{bmatrix}-0.0083\\0.1028\\0.9947\end{bmatrix}$	$\begin{bmatrix}0.7633 & -0.0048 & -0.0451\\0.0058 & 0.7663 & 0.0093\\0.0545 & 0.0431 & 0.9954\end{bmatrix}$	$\begin{bmatrix}0.5412\\0.5400\\0\end{bmatrix}$	44.94°
Coherency matrix	$\frac{1}{2}\begin{bmatrix}1 & 1 & 0\\1 & 1 & 0\\0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0.0485\\0.1217\\0.9914\end{bmatrix}$	$\begin{bmatrix}-0.0083\\0.1028\\0.9947\end{bmatrix}$	$\begin{bmatrix}0.7633 & -0.0048 & -0.0451\\0.0058 & 0.7663 & 0.0093\\0.0545 & 0.0431 & 0.9954\end{bmatrix}$	$\begin{bmatrix}0.2929 & 0.2923 & 0\\0.2923 & 0.2916 & 0\\0 & 0 & 0\end{bmatrix}$	44.94°

5.3 Application to a double Gauss optical lens

We applied the polarization ray tracing method to a double Gauss optical lens with AR coating, using the coherency matrix calculus in three-dimensional global coordinates. The results were compared with those obtained using the two-dimensional calculus. The double Gauss optical lens is shown in Fig. 4.

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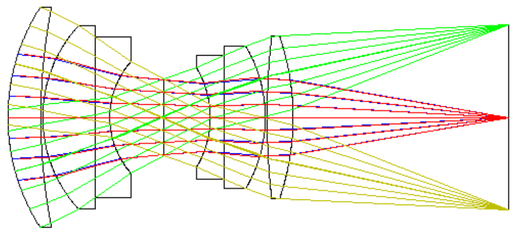

Fig. 4. The double Gauss optical lens. The entrance pupil diameter is 33 mm, the focal length is 100 mm, and the field of view is 28 deg.

We assumed the degree of polarization to be $P = 0.25$, and the polarization orientation $\theta = 25°$, for eigen-vibration coordinates of all light rays. The evolution of the polarization state depended on the normalized pupil coordinates and field coordinates. We performed calculations for all normalized pupil coordinates at some special field coordinates. Figures 5-7 show the results obtained using the two calculation methods. Three-dimensional polarization ray tracing was calculated using the method proposed in this paper, while two-dimensional polarization ray tracing was calculated using the POLDSP.SEQ macro in CODE V according to the Mueller calculus [27].

5.3.1 DoP pupil maps in the central and marginal fields of view

We compared the results obtained using the two methods for the central and marginal fields of view, respectively, to examine the discrepancies across the exit pupil. The normalized field coordinates of the central field of view were (0, 0). The normalized field coordinates of the marginal field of view were (0, 1). As the DoP is an important descriptor of polarization, we constructed DoP pupil maps for the two calculations, to visualize their performance.

The DoP pupil maps for the field coordinates (0, 0) are shown in Fig. 5.

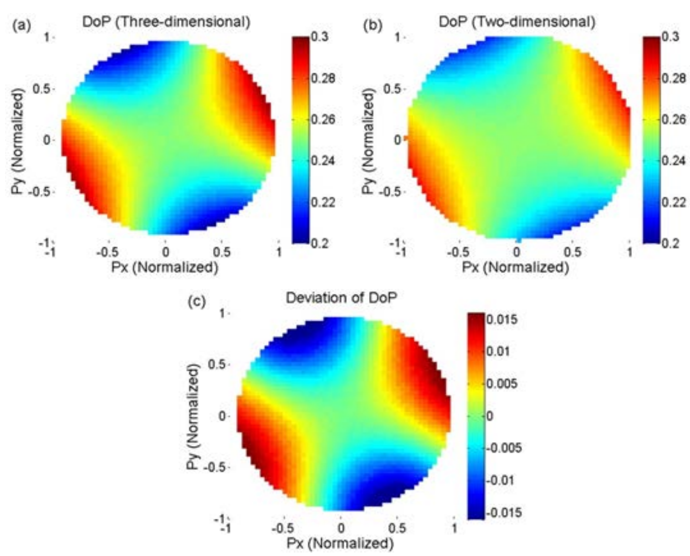

Fig. 5. The DoP pupil maps for the three-dimensional (a) and two-dimensional (b) calculi, and their deviations (c) for the field coordinates (0, 0). The color code corresponds to the values.

Figure 5 shows that the main discrepancy between the two methods is at the edge of two diagonal directions, where the influence of polarization effects is obvious relative to the polarization of the incident light. At the center of the exit pupil, the two methods yield nearly the same results. This is because, in this telephoto lens, the propagation directions of the incident light rays in the central field of view are along the z axis in global coordinates. In the

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cross-shaped area, the discrepancy is small because the directions of polarization effects coincide with the polarization directions of the incident light rays.

The DoP pupil maps for the field coordinates (0, 1) are shown in Fig. 6.

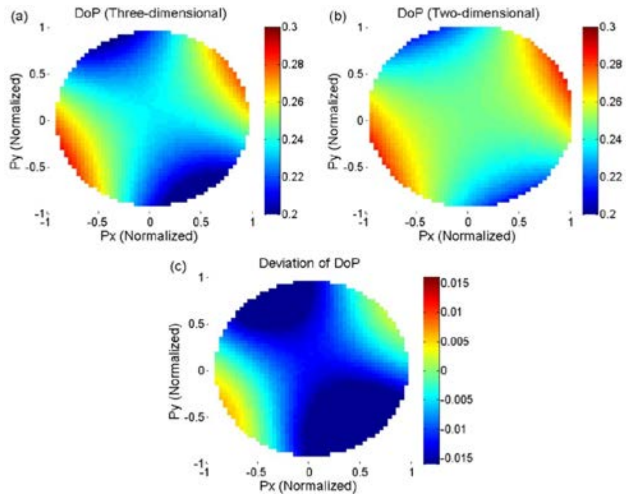

Fig. 6. The DoP pupil maps for the three-dimensional (a) and two-dimensional (b) calculi, and their deviations (c) for the field coordinates (0, 1). The color code corresponds to the values.

Figure 6 shows that the two methods yield different results across the entire exit pupil. This follows because the deviations between the propagation directions of real light rays and their polarization paraxial approximations are much larger in the marginal field of view. Because the two-dimensional calculation assumes the polarization paraxial approximation, many differences between propagation directions are ignored, making the calculation less accurate. On the other hand, these differences are accounted for in the three-dimensional calculation, thus the new method results in more accurate DoP.

5.3.2 The discrepancy between the two calculations across the field of view

To describe the discrepancies across the field of view more explicitly, we averaged the DoP across the pupil to compare the DoP calculated from the two calculi as a function of field coordinates (0, Hy), where Hy represents the normalized field of view coordinates in the *y* direction (Hx = 0). The average DoP and Orient calculated from the two methods are shown in Fig. 7, clearly showing the discrepancies between the two calculation methods.

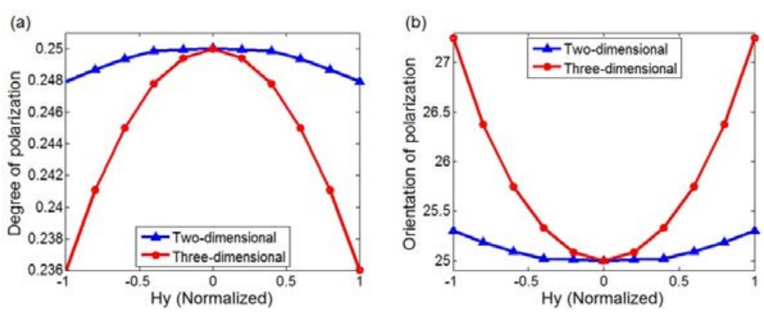

Fig. 7. Average DoP (a) and Orient (b) calculated from the two calculi, vs. the Y field of view. The blue curves with triangles are calculated from the two-dimensional calculation. The red curves with circles are calculated from the three-dimensional calculation.

Figure 7 shows that the three-dimensional calculation suggests larger polarization effect-related changes, compared with the two-dimensional calculation. The discrepancy increases

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from the center field of view to the edges. For this optical system, the largest relative discrepancies for DoP and Orient were ~5% and ~8%, respectively. For optical systems with larger entrance pupil diameters, wider fields of view, or higher numerical apertures, these discrepancies are likely to be more significant.

The above examples demonstrate that the proposed method is more accurate than the two-dimensional calculation method, and has a wider scope of applicability than the generalization of the Jones vector.

6. Conclusions

A framework for polarization ray tracing using 3 × 3 coherency matrices was presented, which allows three-dimensional polarization ray tracing calculations for partially polarized light. The 3 × 3 coherency matrices in three-dimensional coordinate system are proposed to represent both polarized and partially polarized light. The method to calculate the evolution of polarization in three-dimensional-coordinate systems was presented and validated numerically with an example ray trace. The correctness of the proposed method was demonstrated on special examples. The method was demonstrated on a double Gauss optical lens, and the results were compared with those obtained using the conventional two-dimensional calculus. The proposed method is more accurate for non-paraxial ray trace, and very advantageous for optical design, analysis, and polarization calibrations of optical systems that critically rely on polarization.

Funding

National Key R&D Plan of China (2016YFF0103603); National Natural Science Foundation of China (NSFC) (61505199); Program 863 (2011AA12A103).

159.226.165.13 {ts '2018-05-01 21:42:45'}