



The basic research of phase retrieval algorithm

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ABSTRACT

This paper introduces two algorithms of the phase retrieval, Gerchberg–Saxton algorithm and gradient search algorithm. We respectively get the function of the object of gradient search algorithm about the generalized pupil, wavefront and the zernike coefficients of the partial derivatives when double-frame images and their defocus as the input. The relationship between GS algorithm and the gradient search algorithm are revealed. This paper designs the simulation experiment with GS algorithm and gradient search algorithm when single-frame images and double-frame images are used as input. The experiment results show that the gradient search algorithm is superior to GS algorithm for a single-frame image as input. Both GS algorithm and gradient search algorithm can primely work out wavefront for double-frame images of different defocus as the input, but the convergence rate of gradient search algorithm is evidently better than GS algorithm.

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1. Introduction

Phase Retrieval (PR) technology uses diffraction model of light field, gets the intensity distribution of the output surface of the light field by the assumptions of the input light field diffraction calculation. Comparison of the calculated intensity data of output surface of the light field and intensity data of the true generated phase, the minimum error criteria as a rule, through iterative or search to find the most consistent with the phase distribution of real field data. In optics domain, there are many researches on the algorithms of PR, whose core question is Gerchberg–Saxton (GS) algorithm. GS algorithm was first proposed by Gerchberg et al. [1], was subsequently appeared in various algorithms [2–4], so that the PR technology has been widely used, and PR algorithm has become the most important research domain, which is because of its important applications include the wavefront detection [5], X-ray crystallography [6], astronomy [7], transmission electron microscopy and coherent diffractive imaging [8,9], for which $M=2$.

This paper designed the simulation experiment with GS algorithm and gradient search algorithm when single and multiple images were used as input. The experiment results show that the gradient search algorithm is superior to GS algorithm for a single image as input. Both GS algorithm and gradient search algorithm can primely work out wave-front for multi-frame images of different defocus as the input, but the convergence rate of gradient search algorithm is evidently better than GS algorithm. This paper is organized as follows: the theory of PR algorithm is presented in Section 2, the GS algorithm in Section 2.1, the gradient search algorithm in Section 2.2, the relationship between GS algorithm and the gradient search algorithm in Section 2.3, the results and analysis of the simulation experiment in Section 2.4 and the conclusion in Section 3.

2. The theory of PR algorithm

PR is the process of algorithmically finding solutions to the phase problem. PR system is the wavefront sensor of a focal plane waves, a laser spot light on the object plane is a target designated from the focal plane image acquisition, use the acquired image, the defocus of the corresponding image, known pupil size and shape to reverse solve the aberration of the optical system. The structure of the PR system is shown in Fig. 1.

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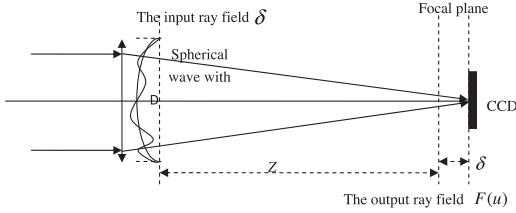


Fig. 1. Schematic of optical path of PR.

Assuming that the aperture of a measured optical system is D , the focal length is Z , the center wavelength of the laser light source is λ , the generalized pupil function for focus plane is $f(x)$, of amplitude $|f(x)|$, and phase $\theta(x)$ (Fig. 2):

$$f(x) = |f(x)| \exp [i\theta(x)], \quad (1)$$

where x is an M -dimensional spatial frequency coordinate, PR consists in finding the phase that for measured amplitude satisfies a set of constraints. And θ is wavefront distortion and can be obtained with Zernike polynomial fitting: $\theta(x) = \sum_n \alpha_n Z_n(x)$, the real number α_n represents the first n terms of polynomial coefficients, Z_n indicates the first n terms of Zernike polynomials basement. For linear optical system, when the generalized pupil $f(x)$ whose defocus is δ in the plane, the impulse response function $F(u)$ is

$$F(u) = |F(u)| \exp [i\psi(u)] = \mathcal{F} \{ f(x) \exp [\varepsilon(x, \delta)] \}, \quad (2)$$

where x is the coordinates of the pupil domain, u is the coordinates of the image domain, both of them are 2-dimensional vector field coordinates. ψ is the phase part of the impulse response, \mathcal{F} 2-dimensional Fourier transform, $\varepsilon(x, \delta)$ is wavefront aberration caused by defocus δ in the position x .

For a PR system, $|f(x)|$ of Eq. (1) is the priority conditions of a known optical system, corresponds to the size and shape of the pupil. $|F(u)|^2$ is the image collected by CCD where the defocus is δ . Therefore, we detect wavefront by PR is to get $\eta(x)$ with the above known quantity.

2.1. GS algorithm

GS algorithm can be described as $g_{m,k}$, $\theta_{m,k}$, $G_{m,k}$, $\phi_{m,k}$, respectively, is the estimate value of f , η , F , ψ when the m th images iterative the k times, g_k represents combine estimate value with every $g_{m,k}$ to f when the k times iterative, which is $g_k(x) = (1/M) \sum_{m=1}^M g_{m,k}(x)$. The steps of GS algorithm are:

Initialization

$$K = 0; \theta_{m,k} = 0, \varepsilon_m(x) = \varepsilon(x, \delta_m) = \left(\frac{\pi \delta_m \|x\|^2}{\lambda Z^2} \right), g_k(x) = |f(x)|, m \in [1, M], \quad (3)$$

$$G_{m,k}(u) = |G_{m,k}(u)| \exp [i\phi_{m,k}(u)] = \mathcal{F} \{ g_k(x) \exp [i\varepsilon_m(x)] \}, m \in [1, M], \quad (4)$$

$$G'_{m,k}(u) = |F(u)| \exp [i\phi_{m,k}(u)], m \in [1, M], \quad (5)$$

$$g'_{m,k}(x) = |g'_{m,k}(x)| \exp [i\theta'_{m,k}(x)] = \mathcal{F}^{-1} [G'_{m,k}(u)] \exp [-\varepsilon_m(x)], m \in [1, M], \quad (6)$$

$$g_{m,k+1}(x) = |f(x)| \exp [i\theta_{m,k+1}(x)] = |f(x)| \exp [i\theta'_{m,k}(x)], m \in [1, M], \quad (7)$$

$$g_{k+1}(x) = \frac{1}{M} \sum_{m=1}^M g_{m,k+1}(x). \quad (8)$$

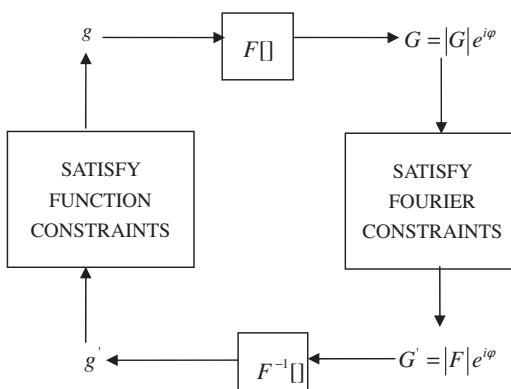


Fig. 2. Block diagram of the Gerchberg-Saxton algorithm.

Repeat from steps b to steps f until extrude the condition, which is the limitations of iterative times or the function of the object descend to appointed value.

The function of the object is

$$B_k = E_{Fk}^2 = N^{-2} \sum_{m=1}^M \sum_u |G_{m,k}(u) - G'_{m,k}(u)|^2, \quad (9)$$

where N represents the width of collected images, those images are foursquare. According to Eqs. (4) and (5), the phase of $G_{m,k}(u)$ and the phase of $G'_{m,k}(u)$ are equal, so Eq. (9) can be:

$$B_k = E_{Fk}^2 = N^{-2} \sum_{m=1}^M \sum_u [|G_{m,k}(u)| - |F(u)|]^2, \quad (10)$$

Schematic 2 view of the GS algorithm for phase retrieval. From the diagram, we can know that GS algorithm can be applied to the questions that both $|F|$ and $|f|$ are known.

2.2. The gradient search algorithm

The gradient search algorithm is another common method to solve the PR problem. It is the application of mathematical optimization method, with the formula (10) as the function of the object, and the unknown quantity about each partial derivative together with the substitution gradient search algorithm, finally obtained estimation of the wavefront distortion corresponding to θ when B_k is smallest.

The most important application of gradient search algorithm is the correct description of the function of the object and its partial derivatives of each variable, we use unknown variables $g(x)$, $\theta(x)$ and α_n of Eq. (1) to get the partial derivative of Eq. (10). We first discuss the partial derivative which is $g(x)$ as unknown variables. Get the derivative from B to $g(x)$, respectively get the derivative from B_k to the real part of ∂g_{real} and imaginary parts of $g(x)$.

$$\begin{aligned} \partial_{g_{\text{real}}} B_k &\equiv \frac{\partial B_k}{\partial g_{\text{real},k}(x)} = 2N^{-2} \sum_{m=1}^M \sum_u [|G_{m,k}(u)| - |F(u)|] \frac{\partial |G_{m,k}(u)|}{\partial g_{\text{real},k}(x)}, \\ \partial_{g_{\text{imag}}} B_k &\equiv \frac{\partial B_k}{\partial g_{\text{imag},k}(x)} = 2N^{-2} \sum_{m=1}^M \sum_u [|G_{m,k}(u)| - |F(u)|] \frac{\partial |G_{m,k}(u)|}{\partial g_{\text{imag},k}(x)} \end{aligned} \quad (11)$$

$$\begin{aligned} \frac{\partial G_{m,k}(u)}{\partial g_{\text{real},k}(x)} &= \frac{\partial}{\partial g_{\text{real},k}(x)} \sum_y g_k(y) \exp[i\varepsilon_m(x)] \exp\left[\frac{-i2\pi uy}{N}\right] = \exp[i\varepsilon_m(x)] \exp\left[\frac{-i2\pi ux}{N}\right] \\ \frac{\partial G_{m,k}(u)}{\partial g_{\text{imag},k}(x)} &= \frac{\partial}{\partial g_{\text{imag},k}(x)} \sum_y g_k(y) \exp[i\varepsilon_m(x)] \exp\left[\frac{-i2\pi uy}{N}\right] = i \exp[i\varepsilon_m(x)] \exp\left[\frac{-i2\pi ux}{N}\right], \end{aligned} \quad (12)$$

$$\frac{\partial |G_{m,k}(u)|}{\partial g_{\text{real},k}(x)} = \frac{\partial [|G_{m,k}(u)|^2]^{1/2}}{\partial g_{\text{real},k}(x)} = \frac{1}{2|G_{m,k}(u)|} \frac{\partial |G_{m,k}(u)|^2}{\partial g_{\text{real},k}(x)} = \frac{G(u) \exp[-i\varepsilon_m(x) + i2\pi ux/N]}{2|G(u)|} + c.c., \quad (13)$$

Eq. (11) changes into:

$$\begin{aligned} \partial_{g_{\text{real}}} B_k &= N^{-2} \sum_{m=1}^M \sum_u [G_{m,k}(u) - |F(u)| G_{m,k}(u) / |G_{m,k}(u)|] = \frac{-iG(u) \exp[-i\varepsilon_m(x) + i2\pi ux/N]}{2|G(u)|} + c.c. \\ \partial_{g_{\text{imag}}} B_k &= -iN^{-2} \sum_{m=1}^M \sum_u [G_{m,k}(u) - |F(u)| G_{m,k}(u) / |G_{m,k}(u)|] = \frac{-iG(u) \exp[-i\varepsilon_m(x) + i2\pi ux/N]}{2|G(u)|} + c.c. \end{aligned} \quad (14)$$

where *c. c.* represents the former plural conjugate. Using Eq. (5) to define $G'_{m,k}(u)$:

$$G'_{m,k}(u) = \frac{|F(u)| G_{m,k}(u)}{|G_{m,k}(u)|} \quad (15)$$

Eq. (14) can be expressed as

$$\begin{aligned} \partial_{g_{\text{real}}} B_k &= 2\text{Real} \sum_m [g_{m,k}(x) - g'_{m,k}(x)], \\ \partial_{g_{\text{imag}}} B_k &= 2\text{Imag} \sum_m [g_{m,k}(x) - g'_{m,k}(x)], \end{aligned} \quad (16)$$

We consider that $\theta(x)$ as the derivative of the unknown value. From Eq. (10), we get the derivative from B_k to $\theta(x)$:

$$\partial_\theta B_k = \frac{\partial B_k}{\partial \theta_k(x)} = 2N^{-2} \sum_m \sum_u [|G_{m,k}(u)| - |F(u)|] \frac{\partial |G_{m,k}(u)|}{\partial \theta_k(x)}, \quad (17)$$

Because of

$$\frac{\partial G_{m,k}(u)}{\partial \theta_k(x)} = \frac{\partial}{\partial \theta_k(x)} \sum_y |f(y)| \exp[i\theta(y)] \exp[i\varepsilon_m(x)] \exp\left[\frac{-i2\pi uy}{N}\right] = ig_k(x) \exp[i\varepsilon_m(x)] \exp\left[\frac{-i2\pi ux}{N}\right], \quad (18)$$

Then get:

$$\frac{\partial |G_{m,k}(u)|}{\partial \theta_k(x)} = \frac{G_{m,k}(u)(-i)g_k^*(x)\exp[-i\varepsilon_m(x)]\exp[i2\pi ux/N] + c.c.}{2|G_{m,k}(u)|}, \quad (19)$$

So we can get:

$$\begin{aligned} \partial_\theta B_k &= \sum_m ig_{m,k}^*(x) [g'_{m,k}(x) - g_{m,k}(x)] + c.c. \\ &= -2\text{Imag} \sum_m [g_{m,k}^*(x)g'_{m,k}(x)], \\ &= -2|f(x)| \sum_m |g'_{m,k}(x)| \sin [\theta'_{m,k}(x) - \theta_{m,k}(x)] \end{aligned}, \quad (20)$$

At last we consider that Zernike coefficient [10,11] $a(x)$ as the derivative of the unknown value. From Eq. (10), we get the derivative from B_k to $a(x)$:

$$\frac{\partial B_k}{\partial a_{n,k}} = \sum_x \frac{\partial B}{\partial \theta_k(x)} \frac{\partial \theta_k(x)}{\partial a_{n,k}(x)}, \quad (21)$$

where

$$\frac{\partial \theta_k(x)}{\partial a_{n,k}} = \frac{\partial}{\partial a_{n,k}} \left[\sum_{n=1}^m a_{n,k} Z_n(x) \right] = Z_n(x), \quad (22)$$

Take Eq. (20) and Eq. (22) into Eq. (21), we get

$$\partial_{a_n} B_k = -2 \sum_m \sum_x |f(x)| |g'_{m,k}(x)| \sin [\theta'_{m,k}(x) - \theta_{m,k}(x)] Z_n(x), \quad (23)$$

2.3. The relationship between GS algorithm and the gradient search algorithm

The method of GS is equivalent to (10) as the direction of steepest descent method Newton of the function of the object, in order to make the problem simple, we have the $M=1$, Eq. (16) can be expressed as

$$\partial_g B = 2 [g(x) - g'(x)], \quad (24)$$

Follow the gradient step length can be determined by the Taylor series expansion of B :

$$B \approx B_k + \sum_x \partial_g B_k [g(x) - g_k(x)], \quad (25)$$

When $g(x)=g_k''(x)$, the first term expansion of B is

$$g_k''(x) - g_k(x) = \frac{-B_k \partial_g B_k}{\sum_y (\partial_g B_k)^2}, \quad (26)$$

We get:

$$\sum_y (\partial_g B_k)^2 = 4 \sum_y [g_k(y) - g'_k(y)]^2 = 4B_k$$

Eq. (26) changes into:

$$g_k''(x) - g_k(x) = -\left(\frac{1}{4}\right) \partial_g B_k = \left(\frac{1}{2}\right) [g'_k(x) - g_k(x)], \quad (27)$$

So, GS method is equivalent to B as the direction of Newton steepest descent method of the function of the object, and the step length is $(1/2) [g'_k(x) - g_k(x)]$. We can predict, for the same target wavefront, using GS algorithm and gradient search algorithm in PR respectively, at the beginning of iteration, the convergence speed of GS algorithm is slightly faster than the gradient search algorithm, but the convergence speed of GS algorithm in the iterative process in later iterations of convergence will be significantly slower than the gradient search algorithm, which is same as respectively using the direction of Newton the steepest direction method and conjugate gradient method for the same problem in an optimization problem should be the same phenomenon [12–18].

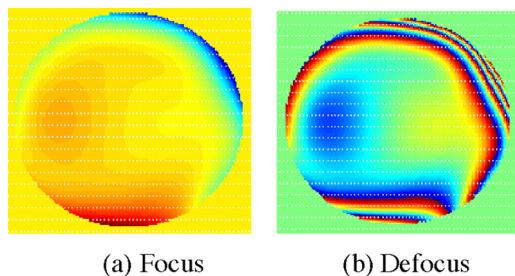


Fig. 3. The simulative wavefront of focus and defocus

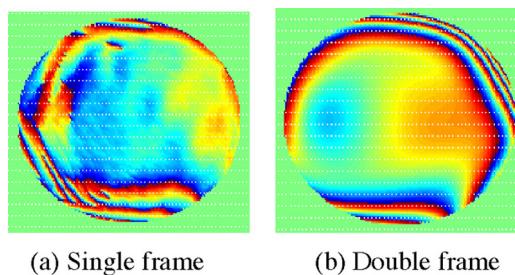


Fig. 4. The results of GS algorithm.

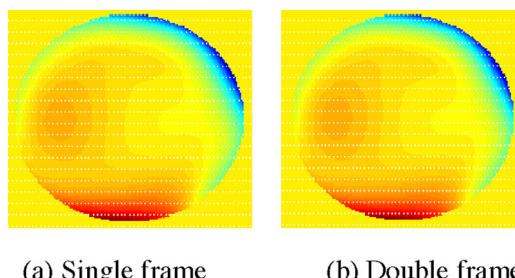


Fig. 5 The results of the gradient search algorithm

2.4. The results and analysis of the simulation experiment

Our simulation of the optical system parameters are: the focal length is 895 mm, the diameter is 20 mm, and the center wavelength of the laser light is 532 nm [19–25]. We generate a target wavefront with RMS = 0.6188, PV = 5.5712, as shown in Fig. 3(a), subtract the integer times of 2π for each point of ω , make it mapped to $(-\pi, \pi]$, get the phase winding form of ω , such as shown in Fig. 3(b). We respectively acquire images at positions focal plane and the defocus amount 1 mm.

2.4.1. The single frame and the double frame of GS Algorithm

We take the collection images of the single frame and double frame into GS algorithm, after 1000 iterations, the results of wavefront solution with single frame is shown in Fig. 4(a), the results of wavefront solution with double frame is shown in Fig. 4(b).

From the simulation experiment we can know, for larger aberration of the target wavefront, GS algorithm using captured image by single frame as the input is difficult to converge to the target wavefront, and using captured image by double frames as an input can get relatively good results.

24.2 The single frame and the double frame of gradient search algorithm

We take the collection images of the single frame and double frame into the gradient search algorithm, the results of wavefront solution with single frame is shown in Fig. 5(a), the results of wavefront solution with double frame is shown in Fig. 5(b).

The RMS wavefront solved by the gradient search algorithm is 0.61188 and PV is 5.5712, both of which are same as the target wavefront.

3 Conclusion

This paper explores the phase retrieval algorithm, especially the GS algorithm and gradient search algorithms are analyzed and compared. We respectively get the function of the object of gradient search algorithm about the generalized pupil, wavefront and the zernike coefficients of the partial derivatives when double-frame images and their defocus as the input. The relationship between GS algorithm and the gradient search algorithm are revealed. This paper designs the simulation experiment with GS algorithm and gradient search algorithm when single-frame images and double-frame images are used as input. The experiment results show that the gradient search algorithm is superior to GS algorithm for a single-frame image as input. Both GS algorithm and gradient search algorithm can prime

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