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A size-dependent four variable refined plate model for functionally graded microplates based on modified couple stress theory

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ABSTRACT

A new size-dependent model for functionally graded microplates is developed by using the modified couple stress theory. In the model, a four variable refined plate theory rather than the first order or any higher order shear deformation theory is adopted to characterize the transverse shear deformation. Firstly, the equations of motion for functionally graded microplates are derived from Hamilton's principle. Then based on these equations, closed-form solutions for bending, buckling and free vibration responses are obtained for simply supported rectangular functionally graded microplates. Furthermore, numerical results based on the analytical solutions are also presented and compared with those predicted by size-dependent first order and third order shear deformation plate models. The results demonstrate that the new size-dependent model has comparable accuracy with the size-dependent third order shear deformation plate model. Thus this new size-dependent model can be easily applied to analyze mechanical responses of functionally graded microplates for its simplicity and high accuracy.

1. Introduction

The concept of functionally graded materials (FGMs) was first introduced in 1984 in Japan as a new kind of composite materials with its properties varying from one surface to another to provide a certain functionality, such as thermal protection from ablation and elimination of stress concentration. Recently, the applications of FGMs have been extended to the micro/nano engineering fields so that functionally graded (FG) micro/nano beams and plates are extensively utilized in atomic force microscopy and micro- and nano-electro-mechanical systems (MEMS and NEMS) [1–4].

In such applications, size effects have been experimentally observed [5–8] and should also play vital roles in potential applications of FG micro/nano beams and plates. Thus, size effects should be taken into account in predicting the static and dynamic responses of FG micro/nano beams and plates. Conventional plate models based on continuum mechanics fail to predict such size effects due to the lack of an intrinsic length scale. In recent years, several size-dependent plate models based on size-dependent continuum theories have been developed, such as nonlocal elasticity theory [9], strain gradient theory [8], couple stress theory [10,11] and its modified version [12]. Among all these theories, the modified couple stress theory has the advantage of involving only one material length scale parameter, thus can be easily used to construct various size-dependent beam and plate models.

Based on the modified couple stress theory, several size-dependent beam models, such as size-dependent Euler–Bernoulli beam model [13,14], Timoshenko beam model [15,16], Reddy–Levinson beam model [17] and other higher order beam models [18,19] have been developed respectively for microbeams. Based on these models, the bending, buckling and free vibration behavior of microbeams were studied by lots of researchers [13–22]. Several size-dependent plate models have also been developed to study the static and dynamic behavior of homogeneous microplates and FG microplates [23–26]. Tsiatas first developed a size-dependent plate model to analyze the bending behavior of microplates by using the classical plate theory (CPT) [23]. This model was then utilized by Yin et al. [27] and Jomehzadeh [28] to study the vibration of microplates. Asghari extended this model to geometrically nonlinear microplates [29] and FG microplates [30]. The CPT accurately predicts results for thin homogeneous microplates, while for moderately thick FG microplates it underestimates the transverse deflection and overestimates the natural frequencies due to neglecting the transverse shear deformation of the plate that is relatively important in this case. To consider the transverse shear deformation, Ma et al. [24] and Ke et al. [31] developed a size-dependent first order model by adopting the first order shear deformation theory (FSDT) rather than the CPT. Recently, Thai and Choi [32] extended the size-dependent first order model to FG

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plates. Although the FSDT predicts sufficiently accurate results for moderately thick FG plates, the necessity of accurately choosing a shear correction factor makes the theory inconvenient to use. Reddy's third-order shear deformation theory (TSDT) takes into account the parabolic variation of the transverse shear strains through the thickness of the plate, thus the transverse shear stresses vanish on the top and bottom surfaces of the plate and a shear correction factor is not required. By employing the TSDT, Thai and Kim [25] developed a size-dependent third-order plate model to study the bending, buckling and free vibration of FG plates. Taking both the transverse shear and normal deformation into account, Reddy and Kim [33] developed a generalized third order plate model, in which more generalized displacements are utilized.

Although the size-dependent third-order plate model provides more accurate results for a moderately thick FG plate without requiring a correction factor, five generalized displacements are needed in the theory to completely characterize the behavior of the plate. Thus when this theory is used to predict the size-dependent behavior of the plate, complex algebraic equations and much computational effort are unavoidable. Recently, Thai and Choi [34] extended a four variable refined plate theory (RPT) that was originally developed for isotropic plates [35] to FG plates to simplify the equations of motion for the shear deformable FG plates. The RPT is variationally consistent, has strong similarity with classical plate theory in many aspects such as equations of motion and boundary conditions, and also assumes parabolic variation of transverse shear stresses through the thickness of the plate without using a shear correction factor. Moreover, the accuracy of the theory has been demonstrated to be comparable to that of the TSDT [34]. Therefore, it is meaningful to extend the RPT to FG microplates by taking the size effects into account.

The purpose of the present paper is to develop a size-dependent refined plate model for FG microplates by using the modified couple stress theory. The equations of motion and boundary conditions for the plates are derived from Hamilton's principle. Analytical solutions for linear bending, buckling and free vibration responses of the plates with simply supported boundary conditions are obtained. Numerical results based on the proposed model are also presented and compared with those predicted by the size-dependent third-order plate model to validate the accuracy of the present model.

2. Theoretical formulation

2.1. Modified couple stress theory

The modified couple stress theory was proposed by Yang et al. [12]. According to this theory, the variation of strain energy can be written as:

$$\delta U = \int_V \sigma_{ij}\delta\varepsilon_{ij}\mathrm{d}V + \int_V m_{ij}\delta\chi_{ij}\mathrm{d}V \tag{1}$$

where the Einstein summation convention is adopted; σ_{ij} are the components of the stress tensor; ε_{ij} are the components of the strain tensor; m_{ij} are the components of the deviatoric part of the symmetric couple stress tensor; and χ_{ij} are the components of the symmetric curvature tensor. The components of the strain and curvature tensors are defined by

$$\begin{aligned}
&\varepsilon_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right) + \frac{1}{2}\frac{\partial u_3}{\partial x_\alpha}\frac{\partial u_3}{\partial x_\beta}, \quad \alpha,\beta = 1,2 \\
&\varepsilon_{3i} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_i} + \frac{\partial u_i}{\partial x_3}\right), \quad i = 1,2,3
\end{aligned} \tag{2}$$

$$\chi_{ij} = \frac{1}{2}\left(\frac{\partial\theta_i}{\partial x_j} + \frac{\partial\theta_j}{\partial x_i}\right), \quad i,j = 1,2,3 \tag{3}$$

where u_i are the components of the displacement vector and θ_i are the components of the rotation vector:

$$\begin{aligned}
&\theta_x = \theta_1 = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right), \quad \theta_y = \theta_2 = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right), \\
&\theta_z = \theta_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)
\end{aligned} \tag{4}$$

2.2. Kinetics

The four variable refined plate theory satisfies the zero traction boundary conditions on the top and bottom surfaces of the plate [34]. Thus, a shear correction factor is not required. The displacement field of this theory is as follows:

$$\begin{aligned}
&u_1(x,y,z,t) = u(x,y,t) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x} \\
&u_2(x,y,z,t) = v(x,y,t) - z\frac{\partial w_b}{\partial y} - f(z)\frac{\partial w_s}{\partial y} \\
&u_3(x,y,z,t) = w(x,y,t) = w_b(x,y,t) + w_s(x,y,t)
\end{aligned} \tag{5}$$

where u and v are respectively the in-plane displacements of a point on the neutral surface of the plate along the x and y coordinates; w_b and w_s are the bending and shear components of transverse displacement u_3 respectively; $f(z)$ is the shape function of shear deformation; and t is the time. It should be noted that $f(z)$ is not unique for many choices are available. In Shimpi's original paper, $f(z)$ was chosen to be $-z/4 + 5z^3/\left(3h^2\right)$ in order to satisfy the assumption that the shear component w_s does not contribute to the bending moment for a plate without gradient properties through its thickness direction [36]. However, the refined plate theory has been extended to the case of FG plates and composite laminated plates where there is no need to make such an assumption. Therefore, $f(z)$ can generally be chosen to other forms of functions as long as zero traction boundary conditions on the top and bottom surfaces are satisfied. Mechab et al. [37] also found such a freedom and pointed out that according to the shape function $\psi(z)$ of any higher order shear deformation plate theory, a refined plate theory can be derived with $f(z) = -\psi(z) + z$. In the present paper, Shimpi's choice is followed.

The strains can be obtained as:

$$\left\{\begin{matrix}\varepsilon_x\\ \varepsilon_x\\ \gamma_{xy}\end{matrix}\right\} = \left\{\begin{matrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{matrix}\right\} + z\left\{\begin{matrix}\kappa_x^b\\ \kappa_y^b\\ \kappa_{xy}^b\end{matrix}\right\} + f\left\{\begin{matrix}\kappa_x^s\\ \kappa_y^s\\ \kappa_{xy}^s\end{matrix}\right\}, \quad \left\{\begin{matrix}\gamma_{yz}\\ \gamma_{xz}\end{matrix}\right\} = g\left\{\begin{matrix}\gamma_{yz}^s\\ \gamma_{xz}^s\end{matrix}\right\} \tag{6}$$

where

$$\left\{\begin{matrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{matrix}\right\} = \left\{\begin{matrix}\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\end{matrix}\right\}, \quad \left\{\begin{matrix}\kappa_x^b\\ \kappa_y^b\\ \kappa_{xy}^b\end{matrix}\right\} = \left\{\begin{matrix}-\frac{\partial^2 w_b}{\partial x^2}\\ -\frac{\partial^2 w_b}{\partial y^2}\\ -2\frac{\partial^2 w_b}{\partial x\partial y}\end{matrix}\right\},$$

$$\left\{\begin{matrix}\kappa_x^s\\ \kappa_y^s\\ \kappa_{xy}^s\end{matrix}\right\} = \left\{\begin{matrix}-\frac{\partial^2 w_s}{\partial x^2}\\ -\frac{\partial^2 w_s}{\partial y^2}\\ -2\frac{\partial^2 w_s}{\partial x\partial y}\end{matrix}\right\},$$

$$\left\{\begin{matrix}\gamma_{yz}^s\\ \gamma_{xz}^s\end{matrix}\right\} = \left\{\begin{matrix}\frac{\partial w_s}{\partial y}\\ \frac{\partial w_s}{\partial x}\end{matrix}\right\}, \quad \text{and} \quad g = 1 - \frac{\mathrm{d}f}{\mathrm{d}z} = \frac{5}{4} - 5\left(\frac{z}{h}\right)^2.$$

Substituting the displacement field from Eq. (5) into Eq. (4), the rotation vector is obtained as:

$$\theta_x = \frac{\partial w_b}{\partial y} + \frac{1}{2}\left(1 + \frac{df}{dz}\right)\frac{\partial w_s}{\partial y}$$
$$\theta_y = -\frac{\partial w_b}{\partial x} - \frac{1}{2}\left(1 + \frac{df}{dz}\right)\frac{\partial w_s}{\partial x} \quad (7)$$
$$\theta_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Substituting Eq. (7) into Eq. (3), the components of the curvature tensor take the following form:

$$\chi_x = \frac{\partial^2 w_b}{\partial x \partial y} + \frac{1}{2}\left(1 + \frac{df}{dz}\right)\frac{\partial^2 w_s}{\partial x \partial y}, \quad \chi_y = -\frac{\partial^2 w_b}{\partial x \partial y} - \frac{1}{2}\left(1 + \frac{df}{dz}\right)\frac{\partial^2 w_s}{\partial x \partial y}, \quad \chi_z = 0,$$
$$\chi_{xy} = \frac{1}{2}\left[\frac{\partial^2 w_b}{\partial y^2} - \frac{\partial^2 w_b}{\partial x^2} + \frac{1}{2}\left(1 + \frac{df}{dz}\right)\left(\frac{\partial^2 w_s}{\partial y^2} - \frac{\partial^2 w_s}{\partial x^2}\right)\right],$$
$$\chi_{yz} = \frac{1}{4}\left[-\frac{d^2 f}{dz^2}\frac{\partial w_s}{\partial x} + \left(\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2}\right)\right],$$
$$\chi_{xz} = \frac{1}{4}\left[\frac{d^2 f}{dz^2}\frac{\partial w_s}{\partial y} + \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y}\right)\right]. \quad (8)$$

2.3. Constitutive equations

Consider a FG microplate made of two constituent functionally graded materials as shown in Fig. 1. The material properties of the plate, such as Young's modulus E, mass density ρ and the Poisson's ratio ν, are assumed to vary continuously through the thickness by a power law as:

$$E(z) = E_2 + (E_1 - E_2)\left(\frac{1}{2} + \frac{z}{h}\right)^p$$
$$\rho(z) = \rho_2 + (\rho_1 - \rho_2)\left(\frac{1}{2} + \frac{z}{h}\right)^p \quad (9)$$
$$\nu(z) = \nu_2 + (\nu_1 - \nu_2)\left(\frac{1}{2} + \frac{z}{h}\right)^p$$

where the subscripts 1 and 2 represent the two materials used, h is the thickness of the plate and p is the pow law index. The linear elastic constitutive relations can be written as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{11} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}, \quad m_{ij} = \frac{E(z) l^2}{1 + \nu(z)}\chi_{ij} \quad (10)$$

where

$$Q_{11} = \frac{E(z)}{1 - \nu(z)^2}, \quad Q_{12} = \frac{E(z)\nu(z)}{1 - \nu(z)^2}, \quad Q_{44} = Q_{55} = Q_{66} = \frac{E(z)}{2[1 + \nu(z)]} = \frac{1}{2}(Q_{11} - Q_{12}),$$

in which l is the material length scale parameter used as a material property to measure the effect of couple stress.

2.4. Equations of motion

Hamilton's principle is utilized to derive the equations of motion. The principle can be stated as:

$$0 = \int_0^T (\delta U + \delta V - \delta K) dt \quad (11)$$

where δU is the variation of strain energy; δV is the variation of potential energy; and δK is the variation of kinetic energy.

The variation of strain energy can be written as:

$$\begin{aligned} \delta U = & \int_V \left(\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_{xy}\delta\gamma_{xy} + \sigma_{yz}\delta\gamma_{yz} + \sigma_{xz}\delta\gamma_{xz}\right) dV \\ & + \int_V \left(m_x \delta\chi_x + m_y \delta\chi_y + 2m_{xy}\delta\chi_{xy} + 2m_{yz}\delta\chi_{yz} + 2m_{xz}\delta\chi_{xz}\right) dV \\ = & \int_A \left[N_x \delta\varepsilon_x^0 + M_x^b \delta\kappa_x^b + M_x^s \delta\kappa_x^s + N_y \delta\varepsilon_y^0 + M_y^b \delta\kappa_y^b + M_y^s \delta\kappa_y^s \right. \\ & \left. + N_{xy}\delta\gamma_{xy}^0 + M_{xy}^b \delta\kappa_{xy}^b + M_{xy}^s \delta\kappa_{xy}^s + Q_{yz}\delta\gamma_{yz}^s + Q_{xz}\delta\gamma_{xz}^s\right] dxdy \\ & + \int_A \left[-\frac{1}{2}P_x \delta\kappa_{xy}^b - \frac{1}{4}R_x \delta\kappa_{xy}^s + \frac{1}{2}P_y \delta\kappa_{xy}^b + \frac{1}{4}R_y \delta\kappa_{xy}^s \right. \\ & - P_{xy}\left(\delta\kappa_y^b - \delta\kappa_x^b\right) - \frac{1}{2}R_{xy}\left(\delta\kappa_y^s - \delta\kappa_x^s\right) - \frac{1}{2}S_{yz}\delta\gamma_{xz}^s \\ & \left. + \frac{1}{2}P_{yz}\delta(v_{,xy} - u_{,yy}) + \frac{1}{2}S_{xz}\delta\gamma_{yz}^s + \frac{1}{2}P_{xz}\delta(v_{,xx} - u_{,xy})\right] dxdy \end{aligned} \quad (12)$$

where N, M^b, M^s, Q, P, R and S are the stress resultants defined by

$$\left(N_i, M_i^b, M_i^s\right) = \int_{-h/2}^{h/2} (1, z, f)\sigma_i dz, \quad i = x, y, xy,$$
$$Q_i = \int_{-h/2}^{h/2} g\sigma_i dz, \quad i = yz, xz$$
$$P_i = \int_{-h/2}^{h/2} m_i dz, \quad i = x, y, xy, yz, xz,$$
$$R_i = \int_{-h/2}^{h/2} m_i\left(1 + \frac{df}{dz}\right) dz, \quad i = x, y, xy,$$
$$S_i = \int_{-h/2}^{h/2} m_i \frac{d^2 f}{dz^2} dz, \quad i = yz, xz.$$

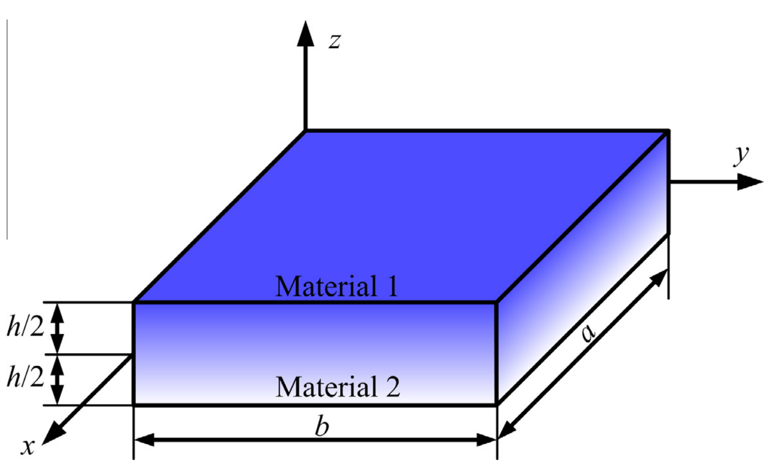

Fig. 1. The geometry of a rectangular FG plate.

Substituting the constitutive equations (10) and Eq. (16) into the above stress resultants, the stress resultants can be expressed in terms of generalized displacements as:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x^b \\ M_y^b \\ M_{xy}^b \\ M_x^s \\ M_y^s \\ M_{xy}^s \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 & C_{11} & C_{12} & 0 \\ A_{12} & A_{11} & 0 & B_{12} & B_{11} & 0 & C_{12} & C_{11} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} & 0 & 0 & C_{66} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 & E_{11} & E_{12} & 0 \\ B_{12} & B_{11} & 0 & D_{12} & D_{11} & 0 & E_{12} & E_{11} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} & 0 & 0 & E_{66} \\ C_{11} & C_{12} & 0 & E_{11} & E_{12} & 0 & F_{11} & F_{12} & 0 \\ C_{12} & C_{11} & 0 & E_{12} & E_{12} & 0 & F_{12} & F_{11} & 0 \\ 0 & 0 & C_{66} & 0 & 0 & E_{66} & 0 & 0 & F_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x^b \\ \kappa_y^b \\ \kappa_{xy}^b \\ \kappa_x^s \\ \kappa_y^s \\ \kappa_{xy}^s \end{Bmatrix}$$

$$\begin{Bmatrix} Q_{yz} \\ Q_{xz} \end{Bmatrix} = \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{xz}^s \end{Bmatrix}$$

$$\begin{aligned}
&P_x = -A_n\kappa_{xy}^b - \frac{1}{2}B_n\kappa_{xy}^s, \quad P_y = A_n\kappa_{xy}^b + \frac{1}{2}B_n\kappa_{xy}^s, \\
&P_{xy} = A_n\left(-\kappa_y^b + \kappa_x^b\right) + \frac{1}{2}B_n\left(-\kappa_y^s + \kappa_x^s\right) \\
&P_{yz} = -\frac{1}{2}C_n\gamma_{xz}^s + \frac{1}{2}A_n\left(v_{,xy} - u_{,yy}\right), \quad P_{xz} = \frac{1}{2}C_n\gamma_{yz}^s + \frac{1}{2}A_n\left(v_{,xx} - u_{,xy}\right), \\
&R_x = -B_n\kappa_{xy}^b - \frac{1}{2}D_n\kappa_{xy}^s, \quad R_y = B_n\kappa_{xy}^b + \frac{1}{2}D_n\kappa_{xy}^s, \\
&R_{xy} = B_n\left(-\kappa_y^b + \kappa_x^b\right) + \frac{1}{2}D_n\left(-\kappa_y^s + \kappa_x^s\right), \\
&S_{yz} = -\frac{1}{2}F_n\gamma_{xz}^s + \frac{1}{2}C_n\left(v_{,xy} - u_{,yy}\right), \quad S_{xz} = \frac{1}{2}F_n\gamma_{yz}^s + \frac{1}{2}C_n\left(v_{,xx} - u_{,xy}\right)
\end{aligned} \tag{13}$$

where

$$(A_i, B_i, C_i, D_i, E_i, F_i) = \int_{-h/2}^{h/2} Q_i\left(1, z, f, z^2, fz, f^2\right)dz, \quad i = 11, 12, 66$$

$$A_{44} = A_{55} = \int_{-h/2}^{h/2} Q_{66}\left(1 - \frac{df}{dz}\right)^2 dz$$

$$(A_n, B_n, C_n, D_n, F_n) = \int_{-h/2}^{h/2} \frac{E(z)l^2}{2[1 + v(z)]}\left(1, 1 + \frac{df}{dz}, \frac{d^2f}{dz^2}, \left(1 + \frac{df}{dz}\right)^2, \left(\frac{d^2f}{dz^2}\right)^2\right)dz$$

The variation of the potential energy can be expressed as [33]:

$$\begin{aligned}
\delta V = &-\left[\int_V \left(\tilde{f}_x\delta u_1 + \tilde{f}_y\delta u_2 + \tilde{f}_z\delta u_3 + \tilde{c}_x\delta\theta_x + \tilde{c}_y\delta\theta_y + \tilde{c}_z\delta\theta_z\right)dV\right. \\
&+ \int_{A^+}\left(q_x^t\delta u_1^t + q_y^t\delta u_2^t + q_z^t\delta u_3^t\right)dxdy + \int_{A^-}\left(q_x^b\delta u_1^b + q_y^b\delta u_2^b + q_z^b\delta u_3^b\right)dxdy \\
&\left.+ \int_\Gamma\int_{-h/2}^{h/2}(\tilde{t}_x\delta u_1 + \tilde{t}_y\delta u_2 + \tilde{t}_z\delta u_3)dzd\Gamma\right] = -\left[\int_A\left(\left(f_x^0 + q_x^0\right)\delta u + \left(f_y^0 + q_y^0\right)\delta v\right.\right. \\
&+ \left(f_z^0 + q_z^0\right)\delta w + \left(c_x^0 - f_y^1 - q_y^1\right)\delta w_{b,y} + \left(\frac{1}{2}c_x^1 - f_y^2 - q_y^2\right)\delta w_{s,y} \\
&\left. - \left(c_y^0 + f_x^1 + q_x^1\right)\delta w_{b,x} - \left(\frac{1}{2}c_y^1 + f_x^2 + q_x^2\right)\delta w_{s,x} + \frac{1}{2}c_z^0\delta\left(v_{,x} - u_{,y}\right)\right)dxdy \\
&\left.+ \int_\Gamma\left(t_x^0\delta u + t_y^0\delta v + t_z^0\delta w - t_y^1\delta w_{b,y} - t_y^2\delta w_{s,y} - t_x^1\delta w_{b,x} - t_x^2\delta w_{s,x}\right)d\Gamma\right]
\end{aligned} \tag{14}$$

where $\tilde{f}_i, \tilde{c}_i, \tilde{t}_i, q_i^t$ and q_i^b $(i = x, y, z)$ are the body forces, the body couples, the surface forces on the side surfaces of the microplate, the distributed forces on the top surface and the distributed forces on the bottom surface respectively, u_i^t and $u_i^b (i = 1, 2, 3)$ are the displacements on the top and bottom surfaces respectively, and

$$\left(f_i^0, f_i^1, f_i^2\right) = \int_{-h/2}^{h/2} \tilde{f}_i(1, z, f)dz,$$

$$(t_i^0, t_i^1, t_i^2) = \int_{-h/2}^{h/2} \tilde{t}_i(1, z, f)dz, \quad q_i^0 = q_i^t + q_i^b$$

$$q_i^1 = h(q_i^t - q_i^b)/2, \quad q_i^2 = f(h/2)(q_i^t - q_i^b),$$

$$(c_i^0, c_i^0) = \int_{-h/2}^{h/2} \tilde{c}_i(1, 1 + df/dz)dz$$

The variation of the kinetic energy is expressed as:

$$\begin{aligned}
\delta K = &\int_A\left(\int_{-h/2}^{h/2}\rho(\dot{u}_1\delta\dot{u}_1 + \dot{u}_2\delta\dot{u}_2 + \dot{u}_3\delta\dot{u}_3)dz\right)dxdy \\
= &\int_A\left[I_0(\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\delta\dot{w}) + I_1\left(-\dot{u}\delta\dot{w}_{b,x} - \dot{w}_{b,x}\delta\dot{u} - \dot{v}\delta\dot{w}_{b,y} - \dot{w}_{b,y}\delta\dot{v}\right)\right. \\
&+ I_2\left(\dot{w}_{b,x}\delta\dot{w}_{b,x} + \dot{w}_{b,y}\delta\dot{w}_{b,y}\right) + I_3\left(-\dot{u}\delta\dot{w}_{s,x} - \dot{w}_{s,x}\delta\dot{u} - \dot{v}\delta\dot{w}_{s,y} - \dot{w}_{s,y}\delta\dot{v}\right) \\
&+ I_4\left(\dot{w}_{b,x}\delta\dot{w}_{s,x} + \dot{w}_{s,x}\delta\dot{w}_{b,x} + \dot{w}_{b,y}\delta\dot{w}_{s,y} + \dot{w}_{s,y}\delta\dot{w}_{b,y}\right) \\
&\left.+ I_5\left(\dot{w}_{s,x}\delta\dot{w}_{s,x} + \dot{w}_{s,y}\delta\dot{w}_{s,y}\right)\right]dxdy
\end{aligned} \tag{15}$$

where $(I_0, I_1, I_2, I_3, I_4, I_5) = \int_{-h/2}^{h/2}\rho(z)\left(1, z, z^2, f, zf, f^2\right)dz$.

Substituting Eqs. (12), (14) and (15) into Eq. (11), integrating by parts, and collecting the coefficients of $\delta u, \delta v, \delta w_b$ and δw_s respectively, the following equations of motion are derived:

$$\begin{aligned}
\delta u : &N_{x,x} + N_{xy,y} + \frac{1}{2}\left(P_{xz,xy} + P_{yz,yy}\right) + f_x^0 + q_x^0 + \frac{1}{2}c_{z,y}^0 \\
&= I_0\ddot{u} - I_1\ddot{w}_{b,x} - I_3\ddot{w}_{s,x}
\end{aligned} \tag{16}$$

$$\begin{aligned}
\delta v : &N_{xy,x} + N_{y,y} - \frac{1}{2}\left(P_{xz,xx} + P_{yz,xy}\right) + f_y^0 + q_y^0 - \frac{1}{2}c_{z,x}^0 \\
&= I_0\ddot{v} - I_1\ddot{w}_{b,y} - I_3\ddot{w}_{s,y}
\end{aligned} \tag{17}$$

$$\begin{aligned}
\delta w_b : &M_{x,xx}^b + 2M_{xy,xy}^b + M_{y,yy}^b + N(w) - P_{x,xy} + P_{y,xy} - P_{xy,yy} \\
&+ P_{xy,xx} + f_z^0 + q_z^0 - c_{x,y}^0 + c_{y,x}^0 + f_{x,x}^1 + f_{y,y}^1 + q_{x,x}^1 + q_{y,y}^1 \\
&= I_0(\ddot{w}_b + \ddot{w}_s) + I_1\left(\ddot{u}_{,x} + \ddot{v}_{,y}\right) - I_2\nabla^2\ddot{w}_b - I_4\nabla^2\ddot{w}_s
\end{aligned} \tag{18}$$

$$\begin{aligned}
\delta w_s : &M_{x,xx}^s + 2M_{xy,xy}^s + M_{y,yy}^s + Q_{xz,x} + Q_{yz,y} + N(w) \\
&+ \frac{1}{2}\left(-R_{x,xy} + R_{y,xy} - R_{xy,yy} + R_{xy,xx}\right) + \frac{1}{2}S_{xz,y} - \frac{1}{2}S_{yz,x} + f_z^0 \\
&+ q_z^0 - \frac{1}{2}c_{x,y}^1 + \frac{1}{2}c_{y,x}^1 + f_{x,x}^2 + f_{y,y}^2 + q_{x,x}^2 + q_{y,y}^2 \\
&= I_0(\ddot{w}_b + \ddot{w}_s) + I_3\left(\ddot{u}_{,x} + \ddot{v}_{,y}\right) - I_4\nabla^2\ddot{w}_b - I_5\nabla^2\ddot{w}_s
\end{aligned} \tag{19}$$

where $N(w) = \left(N_x w_{,x} + N_{xy}w_{,y}\right)_{,x} + \left(N_{xy}w_{,x} + N_y w_{,y}\right)_{,y}$

The boundary conditions of the present theory involve specifying one element of each of the following six pairs:

$$u \quad \text{or} \quad N_u \equiv N_x n_x + N_{xy}n_y + \frac{1}{2}\left(P_{xz,x} + P_{yz,y} + c_z^0\right)n_y \tag{20}$$

$$v \quad \text{or} \quad N_v \equiv N_{xy}n_x + N_y n_y - \frac{1}{2}\left(P_{xz,x} + P_{yz,y} + c_z^0\right)n_x \tag{21}$$

$$u_{,y} \quad \text{or} \quad P_{xz}n_x + P_{yz}n_y \tag{22}$$

$$v_{,x} \quad \text{or} \quad P_{xz}n_x + P_{yz}n_y \tag{23}$$

$$\begin{aligned}
w_b \ \text{or} \ V^b \equiv &\left(M_{x,x}^b + M_{xy,y}^b + P_{xy,x} + P_{y,y} + I_1\ddot{u} - I_2\ddot{w}_{b,x} - I_4\ddot{w}_{s,x} + c_y^0 + f_x^1 + q_x^1\right)n_x \\
&+ \left(M_{xy,x}^b + M_{y,y}^b - P_{x,x} - P_{xy,y} + I_1\ddot{v} - I_2\ddot{w}_{b,y} - I_4\ddot{w}_{s,y} - c_x^0 + f_y^1 + q_y^1\right)n_y \\
&+ \widetilde{N} + \overline{M}_{ns,s}^b
\end{aligned} \tag{24}$$

$$w_{b,n} \quad \text{or} \quad \overline{M}_{nn}^b \tag{25}$$

$$\begin{aligned}
w_s \ \text{or} \ V^s \equiv &\left(M_{x,x}^s + M_{xy,y}^s + \frac{1}{2}R_{xy,x} + \frac{1}{2}R_{y,y} + I_3\ddot{u} - I_4\ddot{w}_{b,x} - I_5\ddot{w}_{s,x}\right. \\
&\left.+ \frac{1}{2}c_y^1 + f_x^2 + q_x^2 + Q_{xz} - \frac{1}{2}S_{yz}\right)n_x \\
&+ \left(M_{xy,x}^s + M_{y,y}^s - \frac{1}{2}R_{x,x} - \frac{1}{2}R_{xy,y} + I_3\ddot{v} - I_4\ddot{w}_{b,y}\right. \\
&\left. - I_5\ddot{w}_{s,y} - \frac{1}{2}c_x^1 + f_y^2 + q_y^2 + Q_{yz} + \frac{1}{2}S_{xz}\right)n_y + \widetilde{N} + \overline{M}_{ns,s}^s
\end{aligned} \tag{26}$$

$$w_{s,n} \quad \text{or} \quad \overline{M}^s_{nn} \tag{27}$$

where

$$\widetilde{N} = (N_x w_{,x} + N_{xy} w_{,y}) n_x + (N_{xy} w_{,x} + N_y w_{,y}) n_y$$
$$\overline{M}^b_{nn} = \left(M^b_x + P_{xy}\right) n_x^2 + \left(M^b_y - P_{xy}\right) n_y^2 + \left(2M^b_{xy} - P_x + P_y\right) n_x n_y$$
$$\overline{M}^b_{ns} = \left(M^b_y - M^b_x - 2P_{xy}\right) n_x n_y + \left(M^b_{xy} - P_x\right) n_x^2 - \left(M^b_{xy} + P_y\right) n_y^2$$
$$\overline{M}^s_{nn} = \left(M^s_x + \frac{1}{2} R_{xy}\right) n_x^2 + \left(M^s_y - \frac{1}{2} R_{xy}\right) n_y^2 + \left(2M^s_{xy} - \frac{1}{2} R_x + \frac{1}{2} R_y\right) n_x n_y$$
$$\overline{M}^s_{ns} = \left(M^s_y - M^s_x - R_{xy}\right) n_x n_y + \left(M^s_{xy} - \frac{1}{2} R_x\right) n_x^2 - \left(M^s_{xy} + \frac{1}{2} R_y\right) n_y^2$$

For FG plate without body force, body couple and tangent tractions q_x and q_y on the bottom and top surfaces of the plate, the corresponding terms in the equations of motion can be discarded. Omitting these terms, differentiating Eqs. (16) and (17) with respect to x and y respectively and adding the two obtained equations, the following three equations are obtained:

$$N_{x,xx} + 2N_{xy,xy} + N_{y,yy} = I_0(\ddot{u}_{,x} + \ddot{v}_{,y}) - I_1 \nabla^2 \ddot{w}_b - I_3 \nabla^2 \ddot{w}_s \tag{28}$$

$$M^b_{x,xx} + 2M^b_{xy,xy} + M^b_{y,yy} - P_{x,xy} + P_{y,xy} - P_{xy,yy} + P_{xy,xx} + N(w) + q_z = I_0(\ddot{w}_b + \ddot{w}_s) + I_1(\ddot{u}_{,x} + \ddot{v}_{,y}) - I_2 \nabla^2 \ddot{w}_b - I_4 \nabla^2 \ddot{w}_s \tag{29}$$

$$M^s_{x,xx} + 2M^s_{xy,xy} + M^s_{y,yy} + Q_{xz,x} + Q_{yz,y} + \frac{1}{2}(-R_{x,xy} + R_{y,xy} - R_{xy,yy} + R_{xy,xx}) + \frac{1}{2} S_{xz,y} - \frac{1}{2} S_{yz,x} + N(w) + q_z = I_0(\ddot{w}_b + \ddot{w}_s) + I_3(\ddot{u}_{,x} + \ddot{v}_{,y}) - I_4 \nabla^2 \ddot{w}_b - I_5 \nabla^2 \ddot{w}_s \tag{30}$$

2.5. Equations of motion in terms of displacements

Substituting the expressions of the stress resultants (13) into Eqs. (28)–(30) yields:

$$A_{11} \nabla^2 \varphi + A_{11} w_{N1} + A_{12} w_{N2} - B_{11} \nabla^4 w_b - C_{11} \nabla^4 w_s = I_0 \ddot{\varphi} - I_1 \nabla^2 \ddot{w}_b - I_3 \nabla^2 \ddot{w}_s \tag{31}$$

$$B_{11} \nabla^2 \varphi + B_{11} w_{N1} + B_{12} w_{N2} - (D_{11} + A_n) \nabla^4 w_b - (E_{11} + B_n/2) \nabla^4 w_s + N(w) + q_z = I_1 \ddot{\varphi} + I_0(\ddot{w}_b + \ddot{w}_s) - I_2 \nabla^2 \ddot{w}_b - I_4 \nabla^2 \ddot{w}_s \tag{32}$$

$$C_{11} \nabla^2 \varphi + C_{11} w_{N1} + C_{12} w_{N2} - (E_{11} + B_n/2) \nabla^4 w_b - (F_{11} + D_n/4) \nabla^4 w_s + (A_{44} + F_n/4) \nabla^2 w_s + N(w) + q_z = I_3 \ddot{\varphi} + I_0(\ddot{w}_b + \ddot{w}_s) - I_4 \nabla^2 \ddot{w}_b - I_5 \nabla^2 \ddot{w}_s \tag{33}$$

where $\varphi = u_{,x} + v_{,y}$, $w_{N1} = \left[\left(w_{,x}^2\right)_{,xx} + \left(w_{,y}^2\right)_{,yy} + 2(w_{,x} w_{,y})_{,xy}\right]/2$, $w_{N2} = w_{,xy}^2 - w_{,xx} w_{,yy}$.

It is evident that accounting for small scale effects by using the modified couple stress theory only leads to an increase of the bending stiffness of the FG plate. When the intrinsic scale is set to zero $(l = 0)$, $A_n = B_n = D_n = F_n = 0$ and the equations of motion will be degenerated to those derived from the four variable shear deformation plate theory.

For linear plate, w_{N1} and w_{N2} are negligible. The in-plane stress resultants N_x, N_y and N_{xy} are also assumed to be constant and equal to externally prescribed stress resultants. In this case the equations of motion are simplified as:

$$A_{11} \nabla^2 \varphi - B_{11} \nabla^4 w_b - C_{11} \nabla^4 w_s = I_0 \ddot{\varphi} - I_1 \nabla^2 \ddot{w}_b - I_3 \nabla^2 \ddot{w}_s \tag{34}$$

$$B_{11} \nabla^2 \varphi - (D_{11} + A_n) \nabla^4 w_b - (E_{11} + B_n/2) \nabla^4 w_s + \overline{N} + q_z = I_1 \ddot{\varphi} + I_0(\ddot{w}_b + \ddot{w}_s) - I_2 \nabla^2 \ddot{w}_b - I_4 \nabla^2 \ddot{w}_s \tag{35}$$

$$C_{11} \nabla^2 \varphi - (E_{11} + B_n/2) \nabla^4 w_b - (F_{11} + D_n/4) \nabla^4 w_s + (A_{44} + F_n/4) \nabla^2 w_s + \overline{N} + q_z = I_3 \ddot{\varphi} + I_0(\ddot{w}_b + \ddot{w}_s) - I_4 \nabla^2 \ddot{w}_b - I_5 \nabla^2 \ddot{w}_s \tag{36}$$

where $\overline{N} = N_x(w_b + w_s)_{,xx} + 2N_{xy}(w_b + w_s)_{,xy} + N_y(w_b + w_s)_{,yy}$.

It can seen that the equations of motion are simplified to three equations with three unknown functions, i.e. φ, w_b and w_s. This group of equations can be used to derive analytical solutions for linear bending, buckling and free vibration of FG microplate.

Subtracting Eq. (34) multiplied by B_{11}/A_{11} from Eq. (35) yields:

$$\overline{D}_{11} \nabla^4 w_b + \overline{E}_{11} \nabla^4 w_s = \overline{N} + q_z - \overline{I}_1 \ddot{\varphi} - I_0(\ddot{w}_b + \ddot{w}_s) + \overline{I}_2 \nabla^2 \ddot{w}_b + \overline{I}_{41} \nabla^2 \ddot{w}_s \tag{37}$$

Subtracting Eq. (34) multiplied by C_{11}/A_{11} from Eq. (36) yields:

$$\overline{E}_{11} \nabla^4 w_b + \overline{F}_{11} \nabla^4 w_s - \overline{A}_{44} \nabla^2 w_s = \overline{N} + q_z - \overline{I}_3 \ddot{\varphi} - I_0(\ddot{w}_b + \ddot{w}_s) + \overline{I}_{42} \nabla^2 \ddot{w}_b + \overline{I}_5 \nabla^2 \ddot{w}_s \tag{38}$$

where

$$\overline{D}_{11} = D_{11} - \frac{B_{11}^2}{A_{11}} + A_n, \quad \overline{E}_{11} = E_{11} - \frac{B_{11} C_{11}}{A_{11}} + \frac{B_n}{2},$$
$$\overline{F}_{11} = F_{11} - \frac{C_{11}^2}{A_{11}} + \frac{D_n}{4}, \quad \overline{A}_{44} = A_{44} + \frac{F_n}{4}$$
$$\overline{I}_1 = I_1 - \frac{B_{11}}{A_{11}} I_0, \quad \overline{I}_2 = I_2 - \frac{B_{11}}{A_{11}} I_1, \quad \overline{I}_{41} = I_4 - \frac{B_{11}}{A_{11}} I_3$$
$$\overline{I}_3 = I_3 - \frac{C_{11}}{A_{11}} I_0, \quad \overline{I}_{42} = I_4 - \frac{C_{11}}{A_{11}} I_1, \quad \overline{I}_5 = I_5 - \frac{C_{11}}{A_{11}} I_3$$

It can be seen from Eqs. (37) and (38) that w_b and w_s are only inertially coupled with the in-plane motion variables u and v. For static bending and buckling problems, the transverse motion is statically decoupled from the in plane motion. When the influence of inertial coupling is relatively small, the inertial coupling term can be directly neglected to derive approximate natural frequencies of the FG microplate.

3. Closed-form solutions for rectangular plates

In the following analysis, bending, buckling and free vibration problems of rectangular plates are considered. Equations (37) and (38) will be used to obtain analytical solutions for the bending deflection and critical buckling load and an approximate formula for natural frequencies of bending vibration with inertial coupling terms in Eqs (37) and (38) being directly neglected. Equations (34), (37) and (38) will also be used to derive an accurate solution for the natural frequencies. Consider a simply supported rectangular plate with length a and width b under transverse load q and in-plane loads in two directions $(N_x = \gamma_1 N, N_y = \gamma_2 N)$.

Following Navier's method, the following series is chosen to represent the displacements which automatically satisfy the simply supported boundary conditions of the plate:

$$w_b = \sum_{m,n=1}^{\infty} W_{bmn} e^{i\omega t} \sin(\alpha x) \sin(\beta y),$$
$$w_s = \sum_{m,n=1}^{\infty} W_{smn} e^{i\omega t} \sin(\alpha x) \sin(\beta y) \tag{39}$$

where $i = \sqrt{-1}$, ω is the natural frequency, $\alpha = m\pi/a, \beta = n\pi/b$, W_{bmn} and W_{smn} are the coefficients. The transverse loads can also be expressed in the following series form:

Table 1
Dimensionless first order natural frequency of a simply supported plate.

a/h	l/h	p = 0		p = 1		p = 10	
		MPT [32]	Present	MPT [32]	Present	MPT [32]	Present
5	0	5.3871	5.3885	4.8744	4.8755	5.5818	5.3793
	0.2	5.7797	5.8439	5.3239	5.3749	5.9551	5.9200
	0.4	6.7996	7.0322	6.4600	6.6512	6.9333	7.2275
	0.6	8.1595	8.6525	7.9298	8.3550	8.2517	8.9116
	0.8	9.6451	10.5052	9.4998	10.2755	9.7045	10.7941
	1	11.1311	12.4874	11.0451	12.3118	11.1666	12.7942
10	0	5.9301	5.9302	5.2697	5.2698	6.1903	6.1157
	0.2	6.3559	6.3770	5.7518	5.7676	6.5967	6.5814
	0.4	7.4807	7.5590	6.9920	7.0532	7.6797	7.7786
	0.6	9.0261	9.1954	8.6477	8.787	9.1829	9.4095
	0.8	10.7848	11.0863	10.4942	10.7543	10.9066	11.2882
	1	12.6360	13.1220	12.4128	12.8483	12.7303	13.3131
20	0	6.0997	6.0997	5.3880	5.3880	6.3837	6.3627
	0.2	6.5376	6.5434	5.8812	5.8854	6.8026	6.7981
	0.4	7.7009	7.7224	7.1571	7.1736	7.9251	7.9523
	0.6	9.3158	9.3625	8.8781	8.9159	9.4993	9.5619
	0.8	11.1801	11.2640	10.8255	10.8968	11.3303	11.4366
	1	13.1786	13.3153	12.8871	13.0076	13.3030	13.4670

$$q(x,y) = \sum_{m,n=1}^{\infty} Q_{mn} \sin(\alpha x) \sin(\beta y) \tag{40}$$

where $Q_{mn} = \frac{4}{ab}\int_0^a \int_0^b q(x,y)\sin(\alpha x)\sin(\beta y)dxdy$. For uniformly distributed transverse load, $Q_{mn} = \frac{4(1-(-1)^m)(1-(-1)^n)q_0}{mn\pi^2}$, and for sinusoidally distributed transverse load, the coefficient $Q_{11} = q_0$ and all other coefficients equal to zero.

Substituting the expansions of w_b, w_s and q into Eqs. (37) and (38), the closed-form solutions can be obtained from the following equations:

$$\left(\begin{bmatrix} s_{11}+k & s_{12}+k \\ s_{12}+k & s_{22}+k \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \right) \begin{Bmatrix} W_{bmn} \\ W_{smn} \end{Bmatrix} = \begin{Bmatrix} Q_{mn} \\ Q_{mn} \end{Bmatrix} \tag{41}$$

where

$$s_{11} = \bar{D}_{11}(\alpha^2+\beta^2)^2, \quad s_{12} = \bar{E}_{11}(\alpha^2+\beta^2)^2,$$
$$s_{22} = \bar{F}_{11}(\alpha^2+\beta^2)^2 + A_{44}(\alpha^2+\beta^2)k = N(\gamma_1\alpha^2+\gamma_2\beta^2),$$
$$m_{11} = I_0 + \bar{I}_2(\alpha^2+\beta^2), \quad m_{12} = I_0 + \bar{I}_{41}(\alpha^2+\beta^2)$$
$$m_{21} = I_0 + \bar{I}_{42}(\alpha^2+\beta^2), \quad m_{22} = I_0 + \bar{I}_5(\alpha^2+\beta^2)$$

For bending analysis, the closed-form solution is obtained by setting the natural frequency and in-plane loads in Eq. (41) equal to zero. The analytical formula for transverse deflection is:

$$w = w_b + w_s = \sum_{m,n=1}^{\infty} \frac{Q_{mn}(s_{11}+s_{22}-2s_{12})}{s_{11}s_{22}-s_{12}^2} \sin(\alpha x)\sin(\beta y) \tag{42}$$

For buckling analysis, the closed-form solution is obtained by setting the natural frequency and the transverse load in Eq. (41) equal to zero. The analytical formula for the critical buckling load is:

$$N_{cr} = \frac{-1}{\gamma_1\alpha^2+\gamma_2\beta^2} \frac{s_{11}s_{22}-s_{12}^2}{s_{11}+s_{22}-2s_{12}} \tag{43}$$

For free vibration analysis, the closed-form solution is obtained by setting the in-plane loads and the transverse load in Eq. (41) equal to zero. The analytical formula for the natural frequency is:

$$\omega^2 = \frac{1}{2(m_{11}m_{22}-m_{12}m_{21})}\Big[s_{11}m_{22}+s_{22}m_{11}-s_{12}(m_{12}+m_{21}) - \sqrt{[s_{11}m_{22}+s_{22}m_{11}-s_{12}(m_{12}+m_{21})]^2 - 4(m_{11}m_{22}-m_{12}m_{21})(s_{11}s_{22}-s_{12}^2)}\Big] \tag{44}$$

In order to obtain the accurate solution for the natural frequency, all the three Eqs. (34), (37) and (38) should be solved simultaneously. Assuming that $\varphi = \sum_{m,n=1}^{\infty} \varphi_{mn} e^{i\omega t} \sin\alpha x \sin\beta y$ and substituting it and Eq. (39) into Eqs. (34), (37) and (38), the following equations for free vibration can be obtained:

$$\left(\begin{bmatrix} s_{11} & s_{12} & 0 \\ s_{12} & s_{22} & 0 \\ s_{31} & s_{32} & s_{33} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \right) \begin{Bmatrix} W_{bmn} \\ W_{smn} \\ \varphi_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{45}$$

where $s_{31} = B_{11}(\alpha^2+\beta^2)^2$, $s_{32} = C_{11}(\alpha^2+\beta^2)^2$, $s_{33} = A_{11}(\alpha^2+\beta^2)$, $m_{13} = \bar{I}_1$, $m_{23} = \bar{I}_3$, $m_{31} = I_1(\alpha^2+\beta^2)$, and $m_{33} = I_3(\alpha^2+\beta^2)$.

Closed-form solution for the above generalized eigenvalue problem can be also derived which is not presented here. Numerical solution is easily obtained by using algorithms for the generalized eigenvalue problem.

Table 2
Dimensionless deflection of a simply supported plate.

l/h	a/h = 5		a/h = 20		a/h = 100	
	TSDT [33]	Present	TSDT [33]	Present	TSDT [33]	Present
0	0.6688	0.6688	0.5689	0.5689	0.5625	0.5625
0.2	0.5468	0.5468	0.4737	0.4737	0.4689	0.4689
0.4	0.3535	0.3535	0.3153	0.3153	0.3128	0.3128
0.6	0.2224	0.2224	0.2025	0.2025	0.2011	0.2011
0.8	0.1464	0.1464	0.1349	0.1349	0.1341	0.1341
1	0.1017	0.1017	0.0944	0.0944	0.0939	0.0939

Table 3
Dimensionless fundamental natural frequency of a simply supported plate.

l/h	a/h = 5			a/h =20			a/h =100		
	TSDT [33]	Pre_ac*	Pre_ap*	TSDT [33]	Pre_ac*	Pre_ap*	TSDT [33]	Pre_ac*	Pre_ap*
0	4.0781	4.0781	4.0895	4.5228	4.5228	4.5239	4.5579	4.5579	4.5580
0.2	4.5094	4.5094	4.5222	4.9568	4.9568	4.9580	4.9922	4.9922	4.9922
0.4	5.6071	5.6071	5.6236	6.0756	6.0756	6.0770	6.1126	6.1126	6.1127
0.6	7.0662	7.0662	7.0880	7.5817	7.5817	7.5836	7.6224	7.6224	7.6224
0.8	8.7058	8.7058	8.7347	9.2887	9.2887	9.2910	9.3344	9.3344	9.3345
1	10.4397	10.4397	10.4780	11.1042	11.1042	11.1070	11.1560	11.1560	11.1561

* Pre_ac denotes accurate results obtained from solving the generalized eigenvalue problem (Eq. (45)), while Pre_ap represents approximate results obtained from Eq. (44).

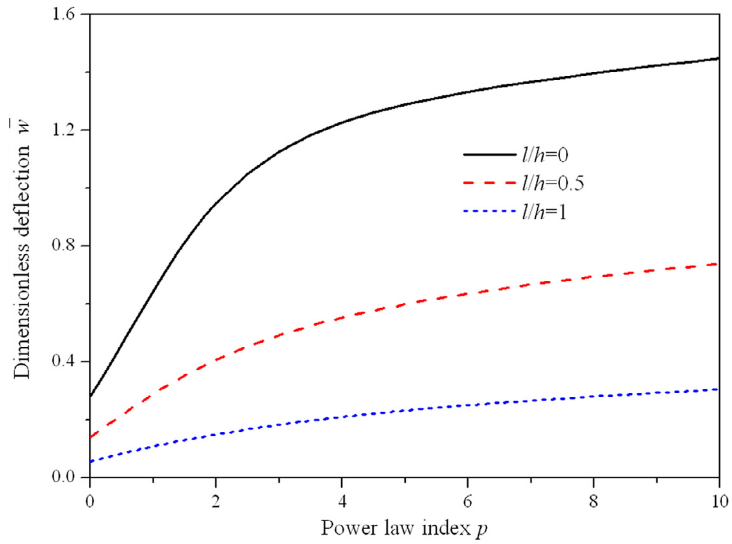

Fig. 2. Effects of the power law index and materials length scale parameter to thickness ratio on the static deflection of FG plate.

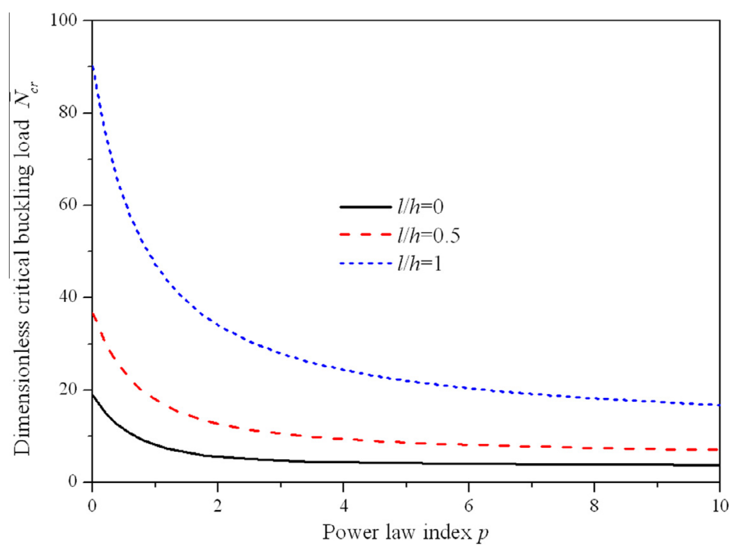

Fig. 3. Effects of the power law index and materials length scale parameter to thickness ratio on the critical buckling load of FG plate.

4. Numerical results and discussion

4.1. Verification studies

In order to verify the correctness and accuracy of the developed theory, results of FG microplates predicted by the model are compared with those reported in the literatures. Firstly consider a simply supported FG microplate with the following material properties [32]:

$$E_1 = 14.4\ \text{GPa},\quad \rho_1 = 12.2 \times 10^3\ \text{kg/m}^3,\quad E_2 = 1.44\ \text{GPa},$$
$$\rho_2 = 1.22 \times 10^3\ \text{kg/m}^3,\quad \nu = 0.38,\quad h = 17.6 \times 10^{-6}\ \text{m},$$
$$q_0 = 1.0\ \text{N/m}$$

The natural frequency is transformed to dimensionless form by using $\bar{\omega} = \omega \frac{a^2}{h}\sqrt{\frac{\rho_1}{E_1}}$. Table 1 lists the dimensionless fundamental natural frequencies for plates with various values of length-to-thickness ratio a/h, materials length scale parameter to thickness ratio l/h and power law index p. The calculated frequencies based on the developed theory are compared with those reported by Thai and Choi [32] based on Mindlin plate theory (MPT). It can be seen that for plates with relatively large $a/h(a/h \geqslant 20)$, relatively small $p(p \leqslant 1)$ and no scale effects, the two theories predict almost the same results. The difference between the frequencies predicted by the two theories respectively increases with the decrease of a/h, because the developed theory captures the shear deformation more accurately than the MPT without using a shear correction factor. Due to the same reason, the difference also increases with the increase of l/h and p.

The developed theory is also compared with the Reddy's third order shear deformation theory (TSDT) with size effects taken into account. Consider a simply supported FG microplate made of aluminum (as material 2) and alumina (as material 1) with the following material properties [33]:

$$E_1 = 380\ \text{GPa},\quad \rho_1 = 3800\ \text{kg/m}^3,\quad E_2 = 70\ \text{GPa},$$
$$\rho_2 = 2702\ \text{kg/m}^3,\quad \nu = 0.3,\quad h = 17.6 \times 10^{-6}\ \text{m},\quad q_0 = 1.0\ \text{N/m}$$

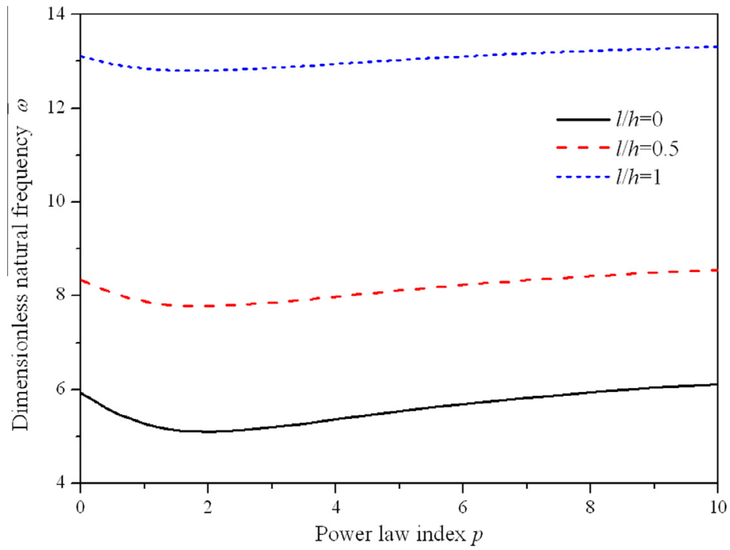

Fig. 4. Effects of the power law index and materials length scale parameter to thickness ratio on the fundamental natural frequency of FG plate.

Table 4
Dimensionless deflection of a simply supported plate.

l/h	a/h =5			a/h =10			a/h =20		
	$p=0$	$p=1$	$p=10$	$p=0$	$p=1$	$p=10$	$p=0$	$p=1$	$p=10$
0	0.3304	0.7384	1.8308	0.2803	0.6472	1.4487	0.2677	0.6243	1.3527
0.2	0.2808	0.6074	1.5069	0.2424	0.5403	1.2505	0.2326	0.5233	1.1850
0.4	0.1937	0.3965	1.0050	0.1725	0.3613	0.8946	0.1670	0.3522	0.8659
0.6	0.1278	0.2512	0.6579	0.1165	0.2328	0.6111	0.1136	0.2280	0.5989
0.8	0.0866	0.1660	0.4470	0.0802	0.1554	0.4245	0.0785	0.1526	0.4186
1	0.0613	0.1156	0.3176	0.0572	0.1089	0.3051	0.0562	0.1071	0.3019

Table 5
Dimensionless critical buckling load of a simply supported plate.

l/h	a/h =5			a/h =10			a/h =20		
	$p=0$	$p=1$	$p=10$	$p=0$	$p=1$	$p=10$	$p=0$	$p=1$	$p=10$
0	15.3322	6.8611	2.7672	18.0754	7.8276	3.4969	18.9243	8.1142	3.7450
0.2	18.0422	8.3399	3.3619	20.9025	9.3767	4.0513	21.7771	9.6815	4.2752
0.4	26.1539	12.7754	5.0407	29.3735	14.0232	5.6631	30.3324	14.3832	5.8505
0.6	39.6393	20.1658	7.7001	43.4732	21.7657	8.2906	44.5855	22.2188	8.4589
0.8	58.4862	30.5105	11.3322	63.1958	32.6036	11.9349	64.5348	33.1882	12.1011
1	82.6938	43.8094	15.9522	88.5416	46.5372	16.6033	90.1804	47.2914	16.7793

For convenience, the following dimensionless forms are used:

$$\bar{w} = \frac{10E_1h^3}{q_0a^4}w\left(\frac{a}{2},\frac{b}{2}\right), \quad \bar{\omega} = \omega\frac{a^2}{h}\sqrt{\frac{\rho_1}{E_1}}$$

For bending analysis, a plate subjected to a sinusoidally distributed load is considered. The closed-form solution can be obtained from Eq. (42). Table 2 lists the dimensionless deflection of plates with various a/h and l/h and constant power law index ($p = 1$). It can be seen that the developed theory predicts almost the same results with those predicted by the TSDT. Thus the accuracy is comparable with that of the TSDT. The same conclusion can also be arrived by comparing fundamental natural frequencies predicted by the developed theory with those predicted by the TSDT as listed in Table 3. The results listed in the table also show that even when a/h is relatively small and l/h is relatively large, approximate values obtained from Eq. (44) match well the accurate results.

4.2. Parameter studies

Consider again the simply supported FG microplate with the following material properties:

$$E_1 = 14.4\ \text{GPa}, \quad \rho_1 = 12.2\times10^3\ \text{kg/m}^3, \quad E_2 = 1.44\ \text{GPa},$$
$$\rho_2 = 1.22\times10^3\ \text{kg/m}^3, \quad \nu = 0.38, \quad h = 17.6\times10^{-6}\ \text{m},$$
$$q_0 = 1.0\ \text{N/m}$$

Parameter studies are presented to investigate the influences of the power law index and material length scale parameter on bending, buckling and free vibration response of the FG microplate. For convenience, the following dimensionless forms are also used:

$$\bar{w} = \frac{100E_2h^3}{q_0a^4}w\left(\frac{a}{2},\frac{b}{2}\right), \quad \overline{N} = \frac{Na^2}{E_2h^3}, \quad \bar{\omega} = \omega\frac{a^2}{h}\sqrt{\frac{\rho_1}{E_1}}$$

As stated by Thai [32], the bifurcation-type of buckling of FG plate with simply supported boundary conditions under in-plane compressive loads will not occur due to the coupling between the in-plane and transverse displacements of the FG plate. For movable-edge plate, the bifurcation-type buckling occurs only when the in-plane loads are applied on the neutral surface. Therefore, in the buckling analysis, the in-plane compressive loads are assumed to be applied on the neutral surface. In this paper, only biaxial buckling problems are considered ($\gamma_1 = \gamma_2 = 1$).

The effects of the power law index and the material length scale parameter on the dimensionless deflection, buckling load, and natural frequency are presented in Figs. 2–4. It can be seen that the deflection decreases with the increase of l/h, while the critical buckling load and fundamental natural frequency increase. This is because the stiffness of the plate increases with the increase of l/h. Moreover, these figures demonstrate that the power law index also has significant effects on the bending, buckling and free vibration behavior. It should be noted that the natural frequency of the plate does not change monotonically with the increase of the power law index. The calculated results for dimensionless deflection and critical buckling load of a simply supported FG plate are also given in Tables 4 and 5.

5. Conclusion

A new size-dependent model for bending, buckling and free vibration of functionally graded microplate is developed by using the modified couple stress theory and Hamilton's principle. The model uses a four variable refined plate theory to characterize the transverse shear deformation and a material length scale parameter to capture the size effects. The refined plate theory has strong similarity with classical plate theory in many aspects and predicts parabolic variation of transverse shear stresses through the thickness of the plate without using a shear correction factor. Closed-form solutions for the bending, buckling and free vibration of functionally graded microplate with simply supported boundary conditions are obtained. Numerical results are also presented and compared with results from size-dependent first order and third order shear deformation theories. The results demonstrate that the new size-dependent model has comparable accuracy with that based on the third order shear deformation theory. Thus the new model can be easily used to analyze mechanical responses of functionally graded microplates due to its simplicity and high accuracy, and it should also be extended to the case of functionally graded microshells.

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