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Stationary distribution of a stochastic SIS epidemic model with vaccination ☆

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HIGHLIGHTS

- It is the first time that stationary distribution for stochastic SISV model and its asymptotic stability are obtained.
- We get the support of the invariant density.
- The solution of stochastic SISV model is ergodic.

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ABSTRACT

In this paper, we consider a stochastic SIS epidemic model with vaccination. We prove that the densities of the distributions of the solution can converge in L^1 to an invariant density under appropriate conditions. Also we find the support of the invariant density.

1. Introduction

In recent years epidemiological modeling of infectious disease transmission has had an increasing influence on the theory and practice of disease management and control. In order to eliminate infectious disease vaccination has been an important strategy. Many authors considered epidemic models with vaccination, see (e.g., Refs. [1–6]). The following model is one of classic SIS models with vaccination:

$$\begin{cases} \dfrac{dS_t}{dt} = A(1-q) - \beta S_t I_t - (\mu + p)S_t + \gamma I_t + \varepsilon V_t, \\ \dfrac{dI_t}{dt} = \beta S_t I_t - (\mu + \gamma + \alpha) I_t, \\ \dfrac{dV_t}{dt} = \mu q + p S_t - (\mu + \varepsilon) V_t. \end{cases} \tag{1.1}$$

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All parameter values in the model are assumed to be nonnegative ($\mu, A > 0$) and summarized in the following list:

A: a constant input of new members into the population per unit time;
q: a fraction of vaccinated for newborns;
β: transmission coefficient between compartments S and I;
p: the proportional coefficient of vaccinated for the susceptible;
μ: the natural death rate of S, I, V compartments;
γ: recovery rate of infectious individuals;
ε: the rate of losing their immunity for vaccinated individuals;
α: disease-caused death rate of infectious individuals.

Due to the existence of environmental noise, the parameters involved in (1.1) are not absolute constants, and they always fluctuate around some average values due to continuous fluctuations in the environment. As a result, the parameters in the model exhibit continuous oscillation around some average values but do not attain fixed values with the advancement of time. In model (1.1) the disease transmission coefficient β is the key parameter to disease transmission. It is of special interest to evaluate the effect of perturbed parameter β on our model. Here we assume that β is subject to the environmental white noise, that is

$$\beta \to \beta + \sigma \dot{B}_t.$$

Consequently, $\beta \mathrm{d}t \to \beta \mathrm{d}t + \sigma \mathrm{d}B_t$, where B_t is a standard Brownian motion, $\sigma^2 > 0$ is the intensity of environmental white noise. Then model (1.1) becomes

$$\begin{cases} \mathrm{d}S_t = [A(1-q) - \beta S_t I_t - (\mu + p)S_t + \gamma I_t + \varepsilon V_t]\mathrm{d}t - \sigma S_t I_t \mathrm{d}B_t, \\ \mathrm{d}I_t = [\beta S_t I_t - (\mu + \gamma + \alpha) I_t]\mathrm{d}t + \sigma S_t I_t \mathrm{d}B_t, \\ \mathrm{d}V_t = (\mu q + p S_t - (\mu + \varepsilon) V_t)\mathrm{d}t. \end{cases} \tag{1.2}$$

The system (1.2) has been considered by Zhao et al. [7]. They obtained that, when the noise is large, the infective decays exponentially to zero regardless of the magnitude of R_0; When the noise is small, sufficient conditions for extinction exponentially and persistence in the mean are established. But in the case of persistence they cannot obtain the existence of stationary distribution of system (1.2). The aim of this paper is to fill the gap. Hence our work can be considered as the further work of Zhao et al. [7].

We assume that $\alpha = 0$ and $A = \mu$. So the total size of the whole population of (1.2) is constant. That is, $S_t + I_t + V_t = 1$. Noting that V_t does not arise explicitly in the first two equations of (1.2), we just need to consider the following two-dimensional system:

$$\begin{cases} \mathrm{d}S_t = [\mu(1-q) - \beta S_t I_t - (\mu + p)S_t + \gamma I_t + \varepsilon(1 - S_t - I_t)]\mathrm{d}t - \sigma S_t I_t \mathrm{d}B_t, \\ \mathrm{d}I_t = [\beta S_t I_t - (\mu + \gamma) I_t]\mathrm{d}t + \sigma S_t I_t \mathrm{d}B_t. \end{cases} \tag{1.3}$$

The deterministic part of system (1.3) has been considered in Ref. [3].

In this paper we are devoted to studying the existence of a stationary distribution of system (1.3) and its asymptotic stability. We will prove that the densities can converge in L^1 to an invariant density under appropriate conditions. Also we find the support of the invariant density.

The difficulty in obtaining stationary distribution derives form the fact that the Fokker–Planck equation corresponding to system (1.3) is of degenerate type, which leads to the invalidity of the approach used in Refs. [8,9]. Here we will employ the Markov semigroup approach [10–12] to obtain the existence of stationary distribution.

Throughout this paper, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathrm{Prob})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_0$ contains all Prob-null sets). Denote

$$\mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}.$$

Since the existence of positive solution of model (1.3) has been obtained by Zhao et al. [7], we take $\mathbb{R}^2_+$ as the whole space. Moreover, it is easy to check that the region $\Gamma^* = \{(x, y) \in \mathbb{R}^2_+ : 0 < x + y < 1\}$ is a positively invariant set of system (1.3). Hence, we always assume that $(S_0, I_0) \in \Gamma^*$.

The rest of this paper is organized as follows. In Section 2, we present our main results and make numerical simulation to support our results. In Section 3, the proof of our results are given. In Section 4, we give a brief conclusion. For the convenience of the reader, in the Appendix we present some auxiliary results concerning Markov semigroups, which contain the main tools used in this paper.

2. Main results and numerical simulation

In this section, we present our result as follows.

Theorem 2.1. *Let (S_t, I_t) be a solution of system (1.3). Then for every $t > 0$ the distribution of (S_t, I_t) has a density $u(t, x, y)$ which satisfies the Fokker–Planck equation (3.1). If $R_0 - 1 > \frac{\sigma^2 m_0^2}{2(\mu+\gamma)}$ and*

$$\frac{a\sigma^2 I^*}{2(\mu+\varepsilon)} < \min\left\{S^{*2}, I^{*2}, (m^0 - S^* - I^*)^2/2, (m_0 - S^* - I^*)^2/2\right\},$$

then there exists a unique density $u_*(x, y)$ which is a stationary solution of (3.1) and

$$\lim_{t\to\infty} \iint_{\mathbb{R}_+^2} |u(t, x, y) - u_*(x, y)| \mathrm{d}x\mathrm{d}y = 0,$$

where

$$R_0 = \frac{\beta m_0}{\mu+\gamma}, \quad m_0 = \frac{\varepsilon + \mu(1-q)}{\mu+\varepsilon+p}, \quad m^0 = \frac{\varepsilon+\mu(1-q)}{\mu+\varepsilon},$$
$$a = \frac{2\mu + 2\varepsilon + p}{\beta}, \quad S^* = \frac{\mu+\gamma}{\beta}, \quad I^* = \frac{(\mu+\gamma)(\mu+\varepsilon+p)}{\beta(\mu+\varepsilon)}(R_0 - 1).$$

In addition, we have

$$\mathrm{supp} u_* = \left\{(x, y) \in \mathbb{R}_+^2 : \ m_0 < x + y < m^0\right\} := E. \tag{2.1}$$

Remark 2.1. By the support of a measurable function f we simply mean the set

$$\mathrm{supp} f = \{(x, y) \in \mathbb{R}_+^2 : f(x, y) \neq 0\}.$$

Remark 2.2. From the result of Theorem 2.1, the solution for system (1.3) is ergodic.

Next we make numerical simulations to illustrate our results by using Milstein's Higher Order Method [13]. We assume that the unit of time is one day and the population sizes are measured in units of 1 million. The parameters in (1.3) are given by

$$\mu = 0.1, \quad p = q = 0.5, \quad \beta = 0.85, \quad \varepsilon = 0.2, \quad \gamma = 0.1, \quad \sigma = 0.5.$$

By direct calculation, we know

$$R_0 - \frac{\sigma^2 m_0^2}{2(\mu+\gamma)} = 1.2671 > 1.$$

We find that these lines in Fig. 1 fit very well which implies that wherever $S(t)$ and $I(t)$ start from, the density functions of $S(t)$ and $I(t)$ converge to the same functions respectively. Fig. 2 indicates that there is a stationary distribution for system (1.3). Hence, Figs. 1 and 2 approve the result of Theorem 2.1. In Fig. 3, blue line and red line are almost the same. This strongly implies ergodicity.

3. Proofs of main result

To prove Theorem 2.1, we need to establish a Markov semigroup connected with model (1.3).

Let $X = \mathbb{R}_+^2$, Σ be the σ-algebra of Borel subsets of X, and m be the Lebesgue measure on (X, Σ). By $\mathscr{P}(t, x, y, A)$ we denote the transition probability function for the diffusion process (S_t, I_t), i.e. $\mathscr{P}(t, x, y, A) = \mathrm{Prob}((S_t, I_t) \in A)$ and (S_t, I_t) is a solution of (1.3) with the initial condition $(S_0, I_0) = (x, y)$. If, for $t > 0$, the distribution of (S_t, I_t) is absolutely continuous with respect to the Lebesgue measure with the density $u(t, x, y)$, then $u(t, x, y)$ satisfies the Fokker–Planck equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sigma^2\left[\frac{\partial^2(x^2y^2u)}{\partial x^2} - 2\frac{\partial^2(x^2y^2u)}{\partial x\partial y} + \frac{\partial^2(x^2y^2u)}{\partial y^2}\right] - \frac{\partial(f_1(x, y)u)}{\partial x} - \frac{\partial(f_2(x, y)u)}{\partial y}, \tag{3.1}$$

where $f_1(x, y) = \mu(1-q) - \beta xy - (\mu+p)x + \gamma y + \varepsilon(1-x-y)$, $f_2(x, y) = \beta xy - (\mu+\gamma)y$.

Now we introduce a Markov semigroup connected with (3.1). Let $P(t)v(x, y) = u(t, x, y)$ for any $v(x, y) \in D$. The definition of D is given in (A.1) (see the Appendix). Since $P(t)$ is a contraction on D, it can be extended to a contraction on $L^1(X, \Sigma, m)$. Thus the operators $\{P(t)\}_{t\geq 0}$ form a Markov semigroup. Let $\mathscr{A}$ be the infinitesimal generator of the semigroup $\{P(t)\}_{t\geq 0}$, i.e.

$$\mathscr{A}v = \frac{1}{2}\sigma^2\left[\frac{\partial^2(x^2y^2v)}{\partial x^2} - 2\frac{\partial^2(x^2y^2v)}{\partial x\partial y} + \frac{\partial^2(x^2y^2v)}{\partial y^2}\right] - \frac{\partial(f_1v)}{\partial x} - \frac{\partial(f_2v)}{\partial y}.$$

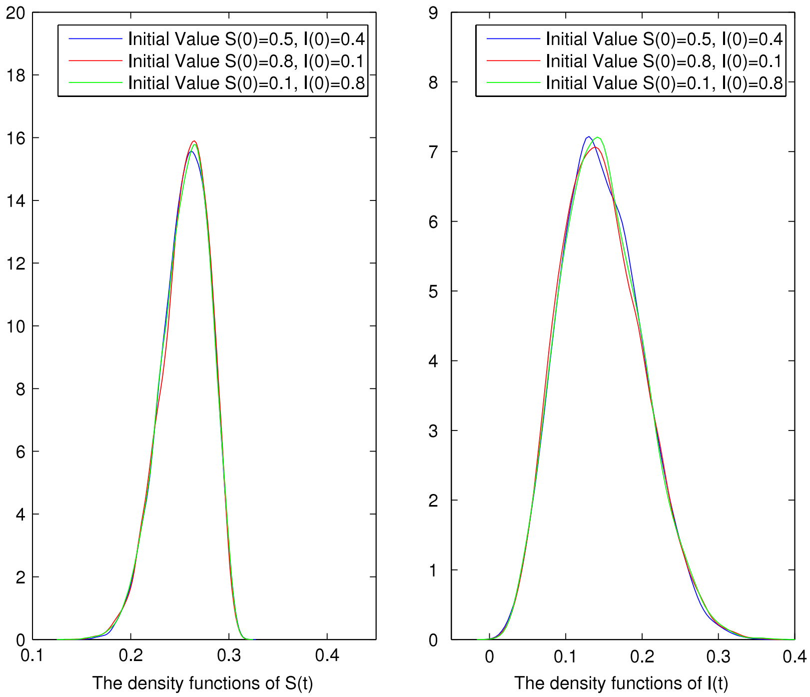

Fig. 1. Based on the 10,000 sample paths, after iterating 10,000 times, we get the density functions of $S(t)$ and $I(t)$ with different initial values. Here $\triangle t = 0.1$.

The adjoint operator of $\mathscr{A}$ is of the form

$$\mathscr{A}^* v = \frac{1}{2}\sigma^2 x^2 y^2 \left[\frac{\partial^2 v}{\partial x^2} - 2\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}\right] + f_1 \frac{\partial v}{\partial x} + f_2 \frac{\partial v}{\partial y}. \tag{3.2}$$

The strategy of the proof of Theorem 2.1 is as follows.

- First, using the Hörmander condition [14] we show that the transition function of the process (S_t, I_t) is absolutely continuous.
- Then, using support theorems [15–17] we prove that the density of the transition function is positive on E (E is given in (2.1)).
- Next, we show that the Markov semigroup satisfies the "Foguel alternative" (see Appendix).
- Finally, we exclude sweeping by showing that there exists a Khasminskiĭ function.

In the following, we realize this strategy by Lemmas 3.1–3.5.

Lemma 3.1. *The transition probability function $\mathscr{P}(t, x_0, y_0, A)$ has a continuous density $k(t, x, y; x_0, y_0)$.*

Proof. If $a(x)$ and $b(x)$ are vector fields on $\mathbb{R}^d$, then the Lie bracket $[a, b]$ is a vector field given by

$$[a, b]_j(x) = \sum_{k=1}^{\mathrm{d}} \left(a_k \frac{\partial b_j}{\partial x_k}(x) - b_k \frac{\partial a_j}{\partial x_k}(x)\right), \quad j = 1, 2, \ldots, d.$$

Let

$$a(\xi, \eta) = \begin{bmatrix} \mu(1-q) - \beta\xi\eta - (\mu+p)\xi + \gamma\eta + \varepsilon(1-\xi-\eta) \\ \beta\xi\eta - (\mu+\gamma)\eta \end{bmatrix}, \qquad b(\xi, \eta) = \begin{bmatrix} -\sigma\xi\eta, \\ \sigma\xi\eta \end{bmatrix}$$

where $(\xi, \eta) \in \mathbb{R}_+^2$. Then by direct calculation,

$$[a, b] = \begin{bmatrix} \sigma\eta[-\mu(1-q) - \varepsilon + (\mu+\varepsilon)\xi + (\varepsilon-\gamma)\eta] \\ \sigma\eta[\mu(1-q) + \varepsilon - (\mu+\varepsilon+p)\xi + (\gamma-\varepsilon)\eta] \end{bmatrix}.$$

Consequently,

$$\begin{vmatrix} -\sigma\xi\eta & \sigma\eta[-\mu(1-q) - \varepsilon + (\mu+\varepsilon)\xi + (\varepsilon-\gamma)\eta] \\ \sigma\xi\eta & \sigma\eta[\mu(1-q) - (\mu+p)\xi + \gamma\eta + \varepsilon - \varepsilon\xi - \varepsilon\eta] \end{vmatrix} = p\sigma\xi^2\eta > 0$$

which means that b, $[a, b]$ are linearly independent on $\mathbb{R}_+^2$.

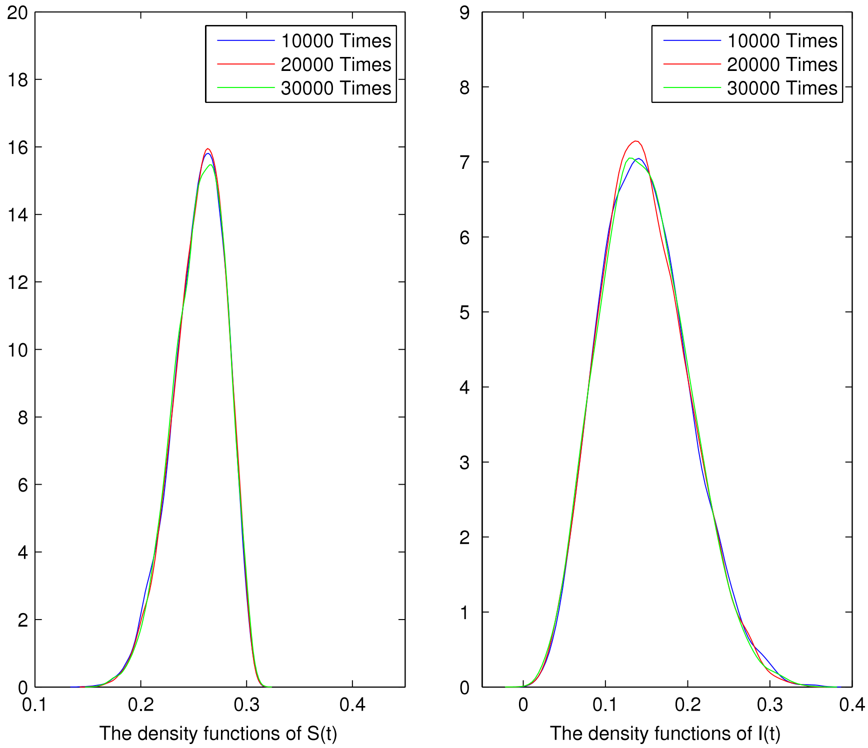

Fig. 2. Based on the 10,000 sample paths, after iterating 10,000 times, 20,000 times and 30,000 times respectively, we get another three groups of density functions of $S(t)$ and $I(t)$ for system (1.3) with $S(0) = 0.5$, $I(0) = 0.3$. Here $\triangle t = 0.1$.

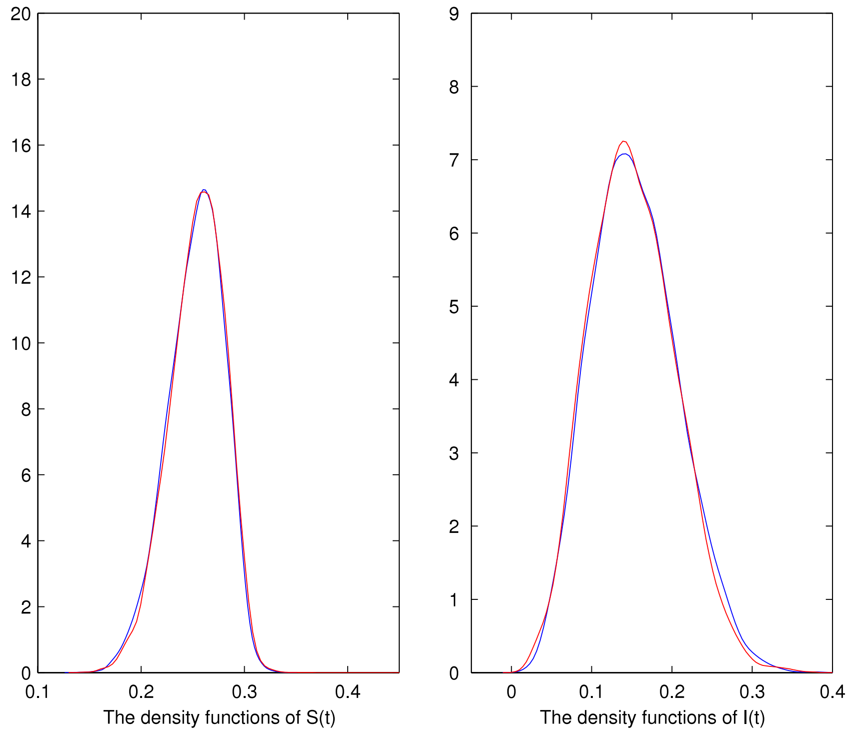

Fig. 3. The blue line corresponds to the density function of the solution for system (1.3) with $S(0) = 0.5$, $I(0) = 0.3$, based on the 10,000 sample paths, after iterating 10,000 times. The red line corresponds to the density function of all states that one trajectory of model (1.3) reaches. Here $\triangle t = 1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Thus for every $(\xi, \eta) \in \mathbb{R}^2_+$, vector $b(\xi, \eta)$, $[a, b](\xi, \eta)$ span the space $\mathbb{R}^2$. In view of Hörmander's Theorem [14], the transition probability function $\mathscr{P}(t, x_0, y_0, A)$ has a density $k(t, x, y; x_0, y_0)$ and $k \in C^\infty\left((0, \infty) \times \mathbb{R}^2_+ \times \mathbb{R}^2_+\right)$. □

Remark 3.1. It follows from Lemma 3.1 that for every $f \in D$,

$$P(t)f(x, y) = \iint_{\mathbb{R}^2_+} k(t, x, y; u, v)f(u, v)\mathrm{d}u\mathrm{d}v.$$

Hence, the semigroup $\{P(t)\}_{t\geq 0}$ is an integral Markov semigroup.

Next, we rewrite SDE (1.3) of Itô type as SDE of Stratonovitch type:

$$\begin{cases} \mathrm{d}S_t = \hat{f}_1(S_t, I_t)\mathrm{d}t - \sigma S_t I_t \circ \mathrm{d}B_t, \\ \mathrm{d}I_t = \hat{f}_2(S_t, I_t)\mathrm{d}t + \sigma S_t I_t \circ \mathrm{d}B_t. \end{cases}$$

where

$$\hat{f}_1(x, y) = \mu(1-q) - \beta xy - (\mu + p)x + \gamma y + \varepsilon(1 - x - y) + \frac{1}{2}\sigma^2[x^2 y - xy^2],$$

$$\hat{f}_2(x, y) = \beta xy - (\mu + \gamma)y - \frac{1}{2}\sigma^2[x^2 y - xy^2].$$

Now we briefly describe the method based on support theorems [15–17] which allows us to check where the kernel k is positive. Fixing a point $(x_0, y_0) \in \mathbb{R}^2_+$ and a function $\phi \in L^2([0, T]; \mathbb{R})$, consider the following system of integral equations:

$$x_\phi(t) = x_0 + \int_0^t [\hat{f}_1(x_\phi(s), y_\phi(s)) - \sigma\phi x_\phi(s) y_\phi(s)]\mathrm{d}s, \tag{3.3}$$

$$y_\phi(t) = y_0 + \int_0^t [\hat{f}_2(x_\phi(s), y_\phi(s)) + \sigma\phi x_\phi(s) y_\phi(s)]\mathrm{d}s. \tag{3.4}$$

Let $D_{x_0,y_0;\phi}$ be the Frechét derivative of the function $h \mapsto \mathbf{x}_{\phi+h}(T)$ from $L^2([0, T]; \mathbb{R})$ to $\mathbb{R}^2$, where $\mathbf{x}_{\phi+h} = \begin{bmatrix} x_{\phi+h} \\ y_{\phi+h} \end{bmatrix}$.

If for some $\phi \in L^2([0, T]; \mathbb{R})$ the derivative $D_{x_0,y_0;\phi}$ has rank 2, then $k(T, x, y; x_0, y_0) > 0$ for $x = x_\phi(T)$ and $y = y_\phi(T)$. The derivative $D_{x_0,y_0;\phi}$ can be found by means of the perturbation method for ODEs. Namely, let $\Gamma(t) = \mathbf{f}'(x_\phi(t), y_\phi(t)) + \mathbf{g}'(x_\phi(t), y_\phi(t))\phi$, where $\mathbf{f}'$ and $\mathbf{g}'$ are the Jacobians of $\mathbf{f} = \begin{bmatrix} \hat{f}_1(x, y) \\ \hat{f}_2(x, y) \end{bmatrix}$ and $\mathbf{g} = \begin{bmatrix} -\sigma xy \\ \sigma xy \end{bmatrix}$ respectively. Let $Q(t, t_0)$, for $T \geq t \geq t_0 \geq 0$, be a matrix function such that $Q(t_0, t_0) = I$, $\partial Q(t, t_0)/\partial t = \Gamma(t)Q(t, t_0)$. Then

$$D_{x_0,y_0;\phi}h = \int_0^T Q(T, s)\mathbf{g}(s)h(s)\mathrm{d}s.$$

Lemma 3.2. *For each $(x_0, y_0) \in E$ and $(x, y) \in E$, there exists $T > 0$ such that $k(T, x, y; x_0, y_0) > 0$, where E is as in (2.1).*

Proof. First, we check that the rank of $D_{x_0,y_0;\phi}$ is 2. Let $\varepsilon \in (0, T)$ and $h(t) = \frac{\mathbf{1}_{[T-\varepsilon,T]}(t)}{x_\phi(t)y_\phi(t)}$, $t \in [0, T]$, where $\mathbf{1}_{[T-\varepsilon,T]}$ is the characteristic function of interval $[T - \varepsilon, T]$. Since $Q(T, s) = I + \Gamma(T)(T - s) + o(T - s)$, we obtain

$$D_{x_0,y_0;\phi}h = \varepsilon\mathbf{v} + \frac{1}{2}\varepsilon^2\Gamma(T)\mathbf{v} + o(\varepsilon^2), \quad \mathbf{v} = \begin{bmatrix} -\sigma \\ \sigma \end{bmatrix},$$

$$\Gamma(T)\mathbf{v} = \begin{bmatrix} \sigma\beta(y - x) + \sigma^2(y - x)\phi + \sigma(\mu + p + \gamma) + \frac{1}{2}\sigma^3[x^2 + y^2] \\ \sigma\beta(x - y) + \sigma^2(x - y)\phi - \sigma(\mu + \gamma) - \frac{1}{2}\sigma^3[x^2 + y^2] \end{bmatrix}.$$

Hence, $\mathbf{v}$ and $\Gamma(T)\mathbf{v}$ are linearly independent. Thus $D_{x_0,y_0;\phi}$ has rank 2.

Next, we prove that for any two points $(x_0, y_0) \in E$ and $(x, y) \in E$, there exist a control function ϕ and $T > 0$ such that $x_\phi(0) = x_0, y_\phi(0) = y_0, x_\phi(T) = x, y_\phi(T) = y$. The system (3.3), (3.4) can be replaced by the following system of differential equations:

$$x'_\phi = \hat{f}_1(x_\phi, y_\phi) - \sigma\phi x_\phi y_\phi, \tag{3.5}$$

$$y'_\phi = \hat{f}_2(x_\phi, y_\phi) + \sigma\phi x_\phi y_\phi. \tag{3.6}$$

Let $z_\phi = y_\phi + x_\phi$. Then (3.5), (3.6) become

$$x'_\phi = g_1(x_\phi, z_\phi) - \sigma\phi x_\phi(z_\phi - x_\phi), \tag{3.7}$$

$$z'_\phi = g_2(x_\phi, z_\phi), \tag{3.8}$$

where

$$g_1(x, z) = \hat{f}_1(x, z - x),$$
$$g_2(x, z) = \varepsilon + \mu(1 - q) - px - (\mu + \varepsilon)z.$$

Let

$$E_0 = \left\{(x, z) \in \mathbb{R}^2_+ : 0 < x < m^0, m_0 < z < m^0 \text{ and } x < z\right\}.$$

Now we claim that for any $(x_0, z_0) \in E_0$ and $(x_1, z_1) \in E_0$ there exist a control function ϕ and $T > 0$ such that $x_\phi(0) = x_0, z_\phi(0) = z_0, x_\phi(T) = x_1$ and $z_\phi(T) = z_1$. If this is the case, it follows that for any two points $(x_0, y_0) \in E$ and $(x, y) \in E$ there exist a control function ϕ and $T > 0$ such that $x_\phi(0) = x_0, y_\phi(0) = y_0, x_\phi(T) = x$ and $y_\phi(T) = y$.

In the following, we will prove that our claim holds. First, we find a positive constant T and a differentiable function

$$z_\phi : [0, T] \to \left(m_0, m^0\right)$$

such that $z_\phi(0) = z_0, z_\phi(T) = z_1, z'_\phi(0) = g_2(x_0, z_0) := z_0^d, z'_\phi(T) = g_2(x_1, z_1) := z_T^d$ and

$$\varepsilon + \mu(1 - q) - (\mu + \varepsilon + p)z_\phi(t) < z'_\phi(t) < \varepsilon + \mu(1 - q) - (\mu + \varepsilon)z_\phi(t), \quad t \in [0, T]. \tag{3.9}$$

We split the construction of the function z_ϕ on three intervals $[0, \tau]$, $[\tau, T - \tau]$ and $[T - \tau, T]$, where $0 < \tau < T/2$. Let

$$\theta = \frac{1}{2}\min\left\{z_0 - m_0, z_1 - m_0, m^0 - z_0, m^0 - z_1\right\}.$$

When $z_\phi \in (m_0 + \theta, m^0 - \theta)$, we have

$$\begin{matrix} \varepsilon + \mu(1 - q) - (\mu + \varepsilon + p)z_\phi(t) < -(\mu + \varepsilon + p)\theta < 0, \\ \varepsilon + \mu(1 - q) - (\mu + \varepsilon)z_\phi(t) > (\mu + \varepsilon)\theta > 0, \end{matrix} \quad t \in [0, T]. \tag{3.10}$$

In view of (3.10) and $z_0 \in (m_0 + \theta, m^0 - \theta)$, we can construct a C^2-function

$$z_\phi : [0, \tau] \to \left(m_0 + \theta, m^0 - \theta\right)$$

such that

$$z_\phi(0) = z_0, \qquad z'_\phi(0) = z_0^{\mathrm{d}}, \qquad z'_\phi(\tau) = 0$$

and z_ϕ satisfies (3.9) for $t \in [0, \tau]$. Analogously, we construct a C^2-function

$$z_\phi : [T - \tau, T] \to \left(m_0 + \theta, m^0 - \theta\right)$$

such that

$$z_\phi(T) = z_1, \qquad z'_\phi(T) = z_T^{\mathrm{d}}, \qquad z'_\phi(T - \tau) = 0$$

and z_ϕ satisfies inequality (3.9) for $t \in [T - \tau, T]$.

Taking T sufficiently large we can extend the function

$$z_\phi : [0, \tau] \cup [T - \tau, T] \to \left(m_0 + \theta, m^0 - \theta\right)$$

to a C^2-function z_ϕ defined on the whole interval $[0, T]$ such that

$$-(\mu + \varepsilon + p)\theta \le z'_\phi(t) \le (\mu + \varepsilon)\theta, \quad t \in [\tau, T - \tau],$$

and therefore, the function z_ϕ satisfies (3.9) on $[0, T]$ in view of (3.10). It follows that we can find a C^1-function x_ϕ which satisfies (3.8) and finally we can determine a continuous function ϕ from (3.7).

Therefore, our claim holds. This completes our proof. □

Lemma 3.3. *Assume $R_0 - 1 > \frac{\sigma^2 m_0^2}{2(\mu+\gamma)}$. For the semigroup $\{P(t)\}_{t\ge0}$ and every density f, we have*

$$\lim_{t\to\infty} \iint_E P(t)f(x, y)\mathrm{d}x\mathrm{d}y = 1,$$

where E is given in (2.1).

Remark 3.2. From Lemmas 3.2 and 3.3, we know that if (3.1) has a stationary solution u_*, then supp$u_* = E$.

Proof. Let $Z_t = S_t + I_t$. The system (1.3) can be replaced by

$$\begin{aligned} dS_t &= [\varepsilon + \mu(1-q) - \beta S_t(Z_t - S_t) - (\mu + p + \gamma)S_t + (\gamma - \varepsilon)Z_t]dt - \sigma S_t(Z_t - S_t)dB_t, \\ dZ_t &= [\varepsilon + \mu(1-q) - pS_t - (\mu + \varepsilon)Z_t]dt. \end{aligned}$$

Since (S_t, I_t) is a positive solution of system (1.3) with probability 1, we get

$$\varepsilon + \mu(1-q) - (\mu + \varepsilon + p)Z_t < \frac{dZ_t}{dt} < \varepsilon + \mu(1-q) - (\mu + \varepsilon)Z_t, \quad t \in (0, +\infty), a.s. \tag{3.11}$$

Now we claim that for almost every $\omega \in \Omega$ there exists $t_0 = t_0(\omega)$ such that

$$m_0 < Z_t(\omega) < m^0 \quad \text{for } t > t_0,$$

which completes our proof. According to the position of initial value Z_0 we consider three cases:

Case 1: $Z_0 \in (m_0, m^0)$. In this case, from (3.11) it is obvious that our claim holds.

Case 2: $Z_0 \in [0, m_0]$. If our claim is false, then we know that there exists $\Omega' \subset \Omega$ with Prob$(\Omega') > 0$ such that $Z_t(\omega) \in [0, m_0]$, $\omega \in \Omega'$. By (3.11), it follows that for any $\omega \in \Omega'$, $Z_t(\omega)$ is strictly increasing on $[0, +\infty)$, and therefore $\lim_{t\to\infty} Z_t(\omega) = m_0$, $\omega \in \Omega'$. From the equation that Z_t satisfies, it follows that $\lim_{t\to\infty} S_t(\omega) = m_0$ and $\lim_{t\to\infty} I_t(\omega) = 0$, $\omega \in \Omega'$. By Itô's Formula, we get

$$d \log I_t = \left(\beta S_t - (\mu + \gamma) - \frac{\sigma^2 S_t^2}{2}\right) dt + \sigma S_t dB_t$$

which yields

$$\frac{\log I_t - \log I_0}{t} = \frac{\beta \int_0^t S_r dr}{t} - (\mu + \gamma) - \frac{\sigma^2 \int_0^t S_r^2 dr}{2t} + \frac{\sigma \int_0^t S_r dB_r}{t}. \tag{3.12}$$

Let $M(t) = \int_0^t S_r dB_r$. Obviously,

$$\limsup_{t\to+\infty} \frac{\langle M, M\rangle_t}{t} = \limsup_{t\to+\infty} \frac{\int_0^t S_r^2 dr}{t} \le 1 < +\infty \quad a.s.$$

By using Strong Law of Large Numbers (Lemma 2.6 in Ref. [18]), we obtain

$$\frac{\int_0^t S_r dB_r}{t} = 0 \quad a.s. \tag{3.13}$$

Condition $R_0 - 1 > \frac{\sigma^2 m_0^2}{2(\mu+\gamma)}$ implies that

$$\begin{aligned} &\lim_{t\to\infty} \left[\frac{\beta \int_0^t S_r dr}{t} - (\mu + \gamma) - \frac{\sigma^2 \int_0^t S_r^2 dr}{2t} + \frac{\sigma \int_0^t S_r dB_r}{t}\right] \\ &\quad = \beta m_0 - (\mu + \gamma) - \frac{\sigma^2 m_0^2}{2} = (\mu + \gamma)\left(R_0 - 1 - \frac{\sigma^2 m_0^2}{2(\mu + \gamma)}\right) > 0, \quad a.s. \text{ on } \Omega'. \end{aligned}$$

By (3.12), we get $\lim_{t\to\infty} \frac{\log I_t}{t} > 0$ $a.s.$ on Ω' which contradicts $\lim_{t\to\infty} I_t(\omega) = 0$, $\omega \in \Omega'$.

Case 3: $Z_0 \in [m^0, 1]$. If our claim is false, by similar arguments to Case 2, we obtain that there exists $\Omega' \subset \Omega$ with Prob$(\Omega') > 0$ such that for any $\omega \in \Omega'$, $\lim_{t\to\infty} Z_t(\omega) = m^0$, $\lim_{t\to\infty} S_t(\omega) = 0$ and $\lim_{t\to\infty} I_t(\omega) = m^0$. Taking $t \to \infty$ in (3.12), we get

$$\lim_{t\to\infty} \frac{\log I_t - \log I_0}{t} = 0, \quad \text{on } \Omega'$$

and

$$\lim_{t\to\infty} \left[\frac{\beta \int_0^t S_r dr}{t} - (\mu + \gamma) - \frac{\sigma^2 \int_0^t S_r^2 dr}{2t} + \frac{\sigma \int_0^t S_r dB_r}{t}\right] = -(\mu + \gamma), \quad a.s. \text{ on } \Omega',$$

where (3.13) is used. This is a contradiction. Thus our claim holds for $Z_0 \in [m^0, 1]$. □

Lemma 3.4. *Assume $R_0 - 1 > \frac{\sigma^2 m_0^2}{2(\mu+\gamma)}$. The semigroup $\{P(t)\}_{t\ge0}$ is asymptotically stable or is sweeping with respect to compact sets.*

Proof. From Lemma 3.1 it follows that $\{P(t)\}_{t\geq 0}$ is an integral Markov semigroup with a continuous kernel $k(t,x,y)$ for $t>0$. From Lemma 3.3 it follows that it is sufficient to investigate the restriction of the semigroup $\{P(t)\}_{t\geq 0}$ to the space $L^1(E)$. According to Lemma 3.2 for every $f\in D$, we have

$$\int_0^\infty P(t)f\,dt>0\quad \text{a.e. on } E.$$

So in view of Lemma A.1, the desired result follows. □

Lemma 3.5. *If* $R_0-1>\frac{\sigma^2 m_0^2}{2(\mu+\gamma)}$ *and*

$$\frac{a\sigma^2 I^*}{2(\mu+\varepsilon)}<\min\left\{S^{*2}, I^{*2}, (m^0-S^*-I^*)^2/2, (m_0-S^*-I^*)^2/2\right\}, \tag{3.14}$$

then the semigroup $\{P(t)\}_{t\geq 0}$ *is asymptotically stable, where* m_0, m^0, S^*, I^*, a *are given in Theorem 2.1.*

Proof. We will construct a nonnegative C^2-function V and a closed set $U\in\Sigma$ (which lies entirely in E) such that

$$\sup_{x\in E\setminus U}\mathscr{A}^*V(x)<0.$$

Such a function is called a Khasminskiĭ function.

Let

$$V(x)=\frac{1}{2}(S-S^*+I-I^*)^2+a\left(I-I^*-I^*\log\frac{I}{I^*}\right),\quad x=(S,I).$$

Then

$$\begin{aligned}\mathscr{A}^*V &= (S-S^*+I-I^*)(\mu(1-q)+\varepsilon-(\mu+\varepsilon+p)S-(\mu+\varepsilon)I)+a(I-I^*)(\beta S-(\mu+\gamma))+\frac{aI^*\sigma^2S^2}{2}\\ &= (S-S^*+I-I^*)((\mu+\varepsilon+p)S^*+(\mu+\varepsilon)I^*-(\mu+\varepsilon+p)S\\ &\quad -(\mu+\varepsilon)I)+a(I-I^*)\beta(S-S^*)+\frac{aI^*\sigma^2S^2}{2}\\ &= (S-S^*+I-I^*)(-(\mu+\varepsilon+p)(S-S^*)-(\mu+\varepsilon)(I-I^*))+a\beta(I-I^*)(S-S^*)+\frac{aI^*\sigma^2S^2}{2}\\ &= -(\mu+\varepsilon+p)(S-S^*)^2-(\mu+\varepsilon)(I-I^*)^2+\frac{aI^*\sigma^2S^2}{2}-(2\mu+2\varepsilon+p-a\beta)(S-S^*)(I-I^*),\end{aligned}$$

where in the second equality we use the fact that (S^*,I^*) is the equilibrium of the deterministic part of system (1.3). In view of $a=(2\mu+2\varepsilon+p)/\beta$, it follows that

$$\begin{aligned}\mathscr{A}^*V &= -(\mu+\varepsilon+p)(S-S^*)^2-(\mu+\varepsilon)(I-I^*)^2+\frac{aI^*\sigma^2S^2}{2}\\ &\leq -(\mu+\varepsilon+p)(S-S^*)^2-(\mu+\varepsilon)(I-I^*)^2+\frac{aI^*\sigma^2}{2}\\ &\leq -(\mu+\varepsilon)(S-S^*)^2-(\mu+\varepsilon)(I-I^*)^2+\frac{aI^*\sigma^2}{2}.\end{aligned}$$

Condition (3.14) implies that the ball

$$-(\mu+\varepsilon)(S-S^*)^2-(\mu+\varepsilon)(I-I^*)^2+\frac{aI^*\sigma^2}{2}=0$$

lies entirely in E. Therefore there exist a closed set $U\in\Sigma$ which contains this ellipsoid and $c>0$ such that

$$\sup_{x\in E\setminus U}\mathscr{A}^*V(x)\leq -c<0.$$

By using similar arguments to those in Ref. [19], the existence of a Khasminskiĭ function implies that the semigroup is not sweeping from the set U. According to Lemma A.1, the semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable, which completes the proof. □

4. Conclusion

This paper studies the existence of stationary distribution of a stochastic SIS epidemic model with vaccination and its asymptotic stability. The proof of our result is based on the techniques developed in Ref. [10]. Rudnicki et al. [10,11] study long-time behavior of a stochastic prey–predator model. They firstly replaced the system under consideration by s slightly simpler one and then proved that for this simpler system there exists a stationary distribution. But this strategy is invalid for our model. This leads to different details which must be made according to specific form of our model. Especially, when constructing a Khasminskiĭ function, Rudnicki et al. [10,11] did not give a explicit expression and those arguments cannot be transferred to our corresponding proof. Here we construct a explicit expression of the Khasminskiĭ function. This is a contribution of this paper.

Appendix

Since the proof of our result is based on the theory of integral Markov semigroups, we need some auxiliary definitions and results concerning Markov semigroups (see Refs. [10,11]). For the convenience of the reader, we present these definitions and results in the Appendix. Let the triple (X, Σ, m) be a σ-finite measure space. Denote by D the subset of the space $L^1 = L^1(X, \Sigma, m)$ which contains all densities, i.e.

$$D = \{f \in L^1 : f \geq 0, \|f\| = 1\}. \tag{A.1}$$

A linear mapping $P : L^1 \to L^1$ is called a Markov operator if $P(D) \subset D$.

The Markov operator P is called an integral or kernel operator if there exists a measurable function $k : X \times X \to [0, \infty)$ such that

$$\int_X k(x, y)m(\mathrm{d}x) = 1 \tag{A.2}$$

for all $y \in X$ and

$$Pf(x) = \int_X k(x, y)f(y)m(\mathrm{d}y)$$

for every density f.

A family $\{P(t)\}_{t\geq 0}$ of Markov operators which satisfies conditions:

(a) $P(0) = \mathrm{Id}$,
(b) $P(t + s) = P(t)P(s)$ for $s, t \geq 0$,
(c) for each $f \in L^1$ the function $t \mapsto P(t)f$ is continuous with respect to the L^1 norm,

is called a Markov semigroup. A Markov semigroup $\{P(t)\}_{t\geq 0}$ is called integral, if for each $t > 0$, the operator $P(t)$ is an integral Markov operator.

We also need two definitions concerning the asymptotic behavior of a Markov semigroup. A density f_* is called invariant if $P(t)f_* = f_*$ for each $t > 0$. The Markov semigroup $\{P(t)\}_{t\geq 0}$ is called asymptotically stable if there is an invariant density f_* such that

$$\lim_{t\to\infty} \|P(t)f - f_*\| = 0 \quad \text{for } f \in D.$$

A Markov semigroup $\{P(t)\}_{t\geq 0}$ is called sweeping with respect to a set $A \in \Sigma$ if for every $f \in D$

$$\lim_{t\to\infty} \int_A P(t)f(x)m(\mathrm{d}x) = 0.$$

We need some result concerning asymptotic stability and sweeping which can be found in Ref. [10] (see Corollary 1).

Lemma A.1. *Let X be a metric space and Σ be the σ-algebra of Borel sets. Let $\{P(t)\}_{t\geq 0}$ be an integral Markov semigroup with a continuous kernel $k(t, x, y)$ for $t > 0$, which satisfies (A.2) for all $y \in X$. We assume that for every $f \in D$ we have*

$$\int_0^\infty P(t)f\,\mathrm{d}t > 0 \quad a.e.$$

Then this semigroup is asymptotically stable or is sweeping with respect to compact sets.

The property that a Markov semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable or sweeping for a sufficiently large family of sets (e.g. for all compact sets) is called the Foguel alternative.

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