Improving the Rapidity of Nonlinear Tracking Differentiator via Feedforward

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***Abstract*—Differential of a signal is required for high-performance motion control and many other fields. Nonlinear tracking differentiator (NTD) is a feasible solution; however, it is limited in achieving better practical effect because of the problem of phase lags. This paper proposes a new idea of design with feedforward. By the feedforward of an input signal in solving the equations of the differentiator, differential estimate is more accurate. The proposal is simple; however, it provides an additional freedom in the design of a differentiator. This idea extends the theoretical structure of NTD. Moreover, it improves the practical effect of the differentiator. The application experiments are implemented in a servo system. Compared to the traditional NTD, the L_2 norm of the servo error is reduced by 88% in a fast motion when the proposal is used. The validity is confirmed.**

***Index Terms*—Differentiator, feedforward, phase lag, rapidity.**

I. Introduction

RECENTLY, the necessity to estimate the time derivative of signals arises in many fields, especially in the field of motion control [1]–[4]. The measurement of velocity and even acceleration is urgently required to implement various control algorithms from classical proportion integration differentiation to modern robust control, sliding mode control, adaptive control, and so on. Usually used velocity sensors and accelerometers are analog devices, which have serious noise and increase the cost.

A high-accuracy optical-electricity encoder provides high-quality position measurement. Therefore, estimating the differential of a position signal is a feasible solution. For optical incremental encoders, the velocity is usually estimated based on two typical concepts: counting the number of pulses during a fixed sampling period (M method) and measuring the time between two pulses (T method). Based on the two methods, researchers proposed various improvements to obtain more accurate velocity estimation [5]–[7]. Practical applications illustrate the validity of these methods. However, more generally, the differential of an absolute signal is also required in a modern motion control system, such as a measurement of any absolute encoder and a received command in a distributed system.

In order to obtain the velocity of a motion system, some model-based approaches were presented, such as the dual-sampling-rate observer and Kalman filter [8], [9]. The required plant model, however, is a limitation in practice. Without any models, it is common in the implementation of a motion control system that a differentiation of a signal r is approximately obtained using the pseudodifferential as

$$y = \frac{gs}{s+g} r \tag{1}$$

which can be rewritten as

$$y = g\left(1 - \frac{g}{s+g}\right) r = g\left(r - \frac{y}{s}\right) \tag{2}$$

where g is the cutoff frequency of a low-pass filter. Equation (2) shows that y/s tracks the input r under a proportional control.

Such approximation is still sensitive to the noise in r because it is amplified by a factor of g. The amplified noise is directly output to y.

To obtain a satisfactory differential estimate, many researchers have tried to develop various differentiators. Levant proposed a differentiator via sliding mode technique [10]. This method obtains the derivative of an input signal and inhibits the influence of signal noise in some degree. However, the information that one needs to know on the signal is an upper bound for Lipschitz's constant of the derivative of the signal. Moreover, the chattering phenomenon is inevitable [11]. Han *et al.* proposed the approach of nonlinear tracking differentiator (NTD) [12], which successfully realized the differential estimate in the presence of noise. This method uses a nonlinear function to realize the signal tracking and makes the problem of calculating a differential turn into solving differential equations by integral. The NTD method has both beautiful theory support in mathematics and good feasibility in engineering. Therefore, the NTD has been studied widely since it came into being [13]. Moreover, the effectiveness has been confirmed by numerous engineering practices [14]–[20].

The original NTD uses an optimal control function of a second-order system with one switching. However, this function is only optimal in the condition of finite acceleration [21]. Therefore, the speed of signal tracking is not satisfactory when the signal changes fast. Because an NTD is a calculation in computer in modern application, the limitation of the acceleration of the virtual second-order system can be very huge. Therefore, to improve the rapidity, other nonlinear functions are

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investigated, such as a finite-time-convergent function [22] and the power function [23].

Previous research usually focuses on the design of the nonlinear function in an NTD to achieve faster and steadier signal tracking and a more accurate differential estimate. However, there are always serious phase lags in the output no matter how the nonlinear function is designed. Due to the phase lags, the accuracy of differential estimate is degraded, which leads to the degradation of the practical effect of an NTD, especially in a high-speed and high-accuracy motion control system.

In this paper, we try to improve the rapidity of an NTD from a different perspective. If the differential estimate is faster and closer to the real value, the accuracy and the practical effect of NTD can be improved. Virtually, the achievement of signal tracking of traditional NTD benefits from nonlinear *feedback*. Therefore, it is considered that constructing *feedforward* should be more effective in improving the rapidity rather than designing the nonlinear function. The NTD with feedforward is proposed. This proposal maintains the advantage of traditional structure and reduces the phase lags effectively. This method provides another freedom in the design, which makes it more flexible. Theoretical analysis is given. Moreover, the practical application in high-accuracy servo control is investigated by experiments, which verifies the validity of the proposal.

This paper is organized as follows. The problem is formulated in Section II. Then, the NTD with feedforward is proposed in Section III. The performance analysis is given in Section IV. The practical application research is carried out in Section V. Finally, this paper is concluded in Section VI.

II. Problem Formulation

To obtain differential approximation, the following system can be utilized:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = u(t). \end{cases} \tag{3}$$

If we select adequate control u to make the state x_1 track an input signal r, the state x_2 is just the approximate differential signal. The input signal including noise is comprised of u. Then, the noise is filtered through an integral calculation from u to x_2. Such a differentiator makes use of the principle of signal tracking and guarantees the quality of the differential.

The general format of a tracking differentiator has been given as (4) [24]

$$\begin{cases} \dot{x}_{1R}(t) = x_{2R}(t),\ x_{1R}(0) = x_{10},\ x_{2R}(0) = x_{20} \\ \dot{x}_{2R}(t) = R^2 f\left[x_{1R}(t) - r(t), \frac{x_{2R}(t)}{R}\right]. \end{cases} \tag{4}$$

In (4), R ($R > 0$) is a real constant gain. x_{1R} and x_{2R} are the states whose changes are relative to R. (x_{10}, x_{20}) is any given initial value. f is a function, which is designed as a nonlinear one to get rapid tracking. An NTD guarantees that, for every $a > 0$, x_{1R} is uniformly convergent to r on $[a, 1)$ as $R \to +\infty$.

However, double integral calculations result in big phase lags in the differential estimate. Moreover, these phase lags are adverse to guaranteeing or improving the system performance when the NTD is applied in a motion control system.

III. Construction of a New Differentiator

To improve the accuracy of estimation, the information of input r is directly introduced into the solution of x_{2R} in (4). Recently, [24] has given the rigorous mathematical proof of a traditional NTD. According to [24, Th. 2.1], an extended design is proposed as the following corollary.

Corollary 3.1: Suppose that the equilibrium point (0, 0) of the following system is globally asymptotically stable:

$$\begin{cases} \dot{z}_1(t) = z_2(t),\ z_1(0) = z_{10} \\ \dot{z}_2(t) = f[z_1(t), z_2(t)],\ z_2(0) = z_{20} \end{cases} \tag{5}$$

where

(z_{10}, z_{20}) is any initial value;

$f$is a locally Lipschitz continuous function, $f(0,0) = 0$.

If the signal r is differentiable and $\sup_{t\in[0,+\infty)} |\dot{r}(t)| < +\infty$, then the solution of the system

$$\begin{cases} \dot{x}_{1R}(t) = x_{2R}(t),\ x_{1R}(0) = x_{10},\ x_{2R}(0) = x_{20} \\ \dot{x}_{2R}(t) = R^2 f\left[x_{1R}(t) - r(t), \frac{x_{2R}(t)}{R}\right] + \alpha\dot{r}(t) \end{cases} \tag{6}$$

is convergent in the sense that, for every $a > 0$, $x_{1R}(t)$ is uniformly convergent to $r(t)$ on $t \in [a, +\infty)$ as $R \to +\infty$, where (x_{10}, x_{20}) is any initial value. $\alpha(\alpha > 0)$ is a constant.

The proof of this corollary can be developed by transforming system (6) into system (5) with a perturbation and using the results in the Appendix that are generalized from [24, Th. 2.1].

Equation set (6) is the new NTD constructed using feedforward. The parameter α is the feedforward gain. This feedforward accelerates the speed of signal tracking between x_{1R} and input r.

The convergence of the NTD is independent of the initial value (x_{10}, x_{20}) according to [24]. They can be arbitrarily selected. If the initial value diverges from the real one, the variable x_{1R} quickly converges on the input signal. Then, the differential estimate also converges on the real value. Normally, the initial value is selected as (0, 0) for convenience.

Remark 3.1: Although the wanted signal differential $\dot{r}(t)$ appears in (6), there is no need for $\dot{r}(t)$ in the implementation of the NTD. The NTD is normally realized in a computer as the following sequences. First, one should calculate $x_{2R}(t)$ as (7) using old values of x_{1R} and x_{2R} (in the first step of integration, the old values are the initial ones)

$$\begin{aligned} x_{2R}(t) = &\int_0^t \left\{R^2 f\left[x_{1R}(\tau) - r(\tau), \frac{x_{2R}(\tau)}{R}\right]\right. \\ &\left. + \alpha\dot{r}(\tau)\right\} d\tau + x_{20} \\ = &R^2 \int_0^t f\left[x_{1R}(\tau) - r(\tau), \frac{x_{2R}(\tau)}{R}\right] d\tau \\ &+ \alpha[r(t) - r(0)] + x_{20}. \end{aligned} \tag{7}$$

Then, calculate x_{1R} by integrating x_{2R}. According to the aforementioned process, the calculation can continue using only the input signal r and internal variables x_{1R} and x_{2R}.

Remark 3.2: To improve the rapidity of signal tracking, the noise in approximative differential must be amplified in some degree because the rapidity and noise suppression are contradictory. However, the noise injected into x_{2R} is only amplified by the feedforward gain α that is an independent freedom in design. It can be selected flexibly and not excessively. Then, the advantage of NTD is still maintained, which makes use of the numerical integration instead of direct differential. Therefore, the proposed approach can accelerate the differential estimate with little cost in noise tolerance.

Remark 3.3: Notice that the format in (6) is the most fundamental design of the proposal. Based on this, a lot of developments can be obtained. For example, a low-pass filter can be utilized in the feedforward, which improves the performance of differential estimation in low frequency and maintains better noise tolerance.

In practice, the nonlinear function f can be selected as any local Lipschitz continuous function. In this paper, a specific design is given using a combination of the power function $f(z) = (\beta z)^{p/q} + z$ as

$$f(z_1, z_2) = -\alpha_1 \left[(\beta z_1)^{\frac{p}{q}} + z_1\right] - \alpha_2 \left[(\beta z_2)^{\frac{p}{q}} + z_2\right] \quad (8)$$

where z_1 and z_2 are the arguments of the function; $\alpha_1(\alpha_1 > 0)$ and $\alpha_2(\alpha_2 > 0)$ are, respectively, the weight of z_1 and z_2; $\beta(\beta \geq 1)$ is the weight between linear and nonlinear characteristics; and p and $q(p > q > 0)$ are both odd numbers which decide the shape of f. Using power function (8), the proposed NTD is written as (9), shown at the bottom of the page. As the aforementioned Remark 3.3, a low-pass filter can be used with α when the differentiator is applied in a system with small signal-to-noise ratio. In this case, $\alpha g/s + g$ is used in the feedforward instead of a constant gain α in the implementation (7)

Fig. 1 illustrates the shapes of the power functions with different values of p/q. According to the rates of the slope, the value of p/q is bigger, and the speed of convergence is faster when the argument is away from equilibrium point 0. Correspondingly, the speed of convergence is slower when the argument is near the equilibrium point. Therefore, great p/q is suitable for the situation where the input signal has a small magnitude. Small p/q is suitable for the input signal with large magnitude. Two parameters p and q provide more options than an integer. Notice that the convergence is guaranteed whatever p and q are selected. Therefore, in practical engineering, the parameters can be designed by fixing q and tuning p using the method of trial and error until satisfactory accuracy is achieved.

The parameter β decides the range of the linear parts in the NTD. When $\beta = 0$, the nonlinear function turns to a linear function. If β is bigger, the linear range is narrower, and the nonlinear range is wider. However, excessive β causes fluctuation in an NTD because the speed of convergence changes in the nonlinear range. Therefore, to efficiently take advantage of a nonlinear characteristic, the parameter β can be selected as great as possible by trial and error based on the premise that fluctuation does not occur in the output of an NTD.

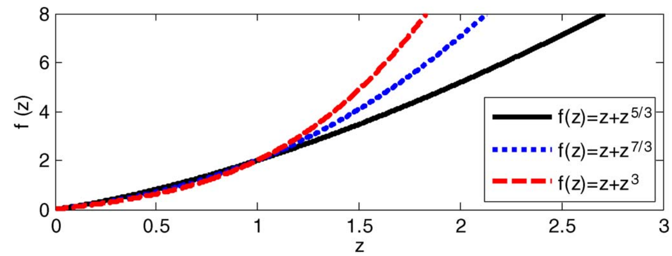

Fig. 1. Shapes of different power functions.

The other parameters can be designed according to the characteristics of a linear system by assuming the nonlinear function (8) as a linear one $f(z_1, z_2) = -\alpha_1 k z_1 - \alpha_2 k z_2, \sim (k > 0)$ at any point (z_1, z_2) with very small neighborhood. Then, to achieve enough damping, α_1 and α_2 should meet $\alpha_2 \geq 2\sqrt{\alpha_1}$. When $\alpha_2 = 2\sqrt{\alpha_1}$, the damping ratio is equal to one. R decides the natural frequency, which should be greater than the radian frequency of the input signal. It is well known that excessive feedforward gain also leads to an overshoot. In practice, α can be designed between 0 and $R\alpha_2$ to decrease phase lag effectively without causing large overshoot.

IV. Analysis

This section analyzes the performance of the proposed and traditional NTDs in frequency domain. It is very difficult to evaluate the frequency characteristic of a nonlinear system directly. Therefore, approximate analysis is employed in this section [25].

First, suppose that the describing function of the nonlinear function f is $N(A)$ when the input is a sinusoid $A\sin(\omega t)$

$$N(A) = \frac{B_1 + jA_1}{A} \quad (10)$$

where A_1 and B_1 are the first-order Fourier coefficient:

$$A_1 = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(\omega t) d\omega t \quad (11)$$

$$B_1 = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(\omega t) d\omega t. \quad (12)$$

Because a real signal and its differential are always real numbers, A_1 is 0, and the nonlinear function f should be an odd

$$\begin{cases} \dot{x}_{1R}(t) = x_{2R}(t), x_{1R}(0) = x_{10}, \ x_{2R}(0) = x_{20} \\ \dot{x}_{2R}(t) = -R^2_{\alpha 1} \left\{ \beta^{\frac{p}{q}} [x_{1R}(t) - r(t)]^{\frac{p}{q}} + [x_{1R}(t) - r(t) \right\} - R^2_{\alpha 2} \left\{ \beta^{\frac{p}{q}} \left[\frac{x_{2R}(t)}{R}\right]^{\frac{p}{q}} + \frac{x_{2R}(t)}{R} \right\} + \alpha \dot{r}(t) \end{cases} \quad (9)$$

function. Without losing generality, mark the approximative linear format of the traditional NTD as

$$\begin{cases} \dot{x}_{1R}(t) = x_{2R}(t),\ x_{1R}(0) = x_{10},\ x_{2R}(0) = x_{20} \\ \dot{x}_{2R}(t) = R^2 \left\{ P\left[x_{1R}(t) - r(t)\right] + \frac{Q}{R} x_{2R}(t) \right\} \end{cases} \tag{13}$$

where P and Q are the functions of A and B_1.

Then, the approximative linear transfer function $G_{RX_{2R}}$ of the traditional NTD from r to x_{2R} is calculated as

$$G_{RX_{2R}}(s) = \frac{-R^2 P s}{s^2 - RQs - R^2 P}. \tag{14}$$

Therefore, to guarantee the stability and correct differential estimate, P and Q should meet $P < 0$ and $Q < 0$. The magnitude-frequency and phase-frequency characteristics are obtained as (15) and (16), respectively

$$|G_{RX_{2R}}(j\omega)| = \frac{-R^2 P\omega}{\sqrt{(\omega^2 + R^2 P)^2 + R^2 Q^2 \omega^2}} \tag{15}$$

$$\angle G_{RX_{2R}}(j\omega) = \begin{cases} \frac{\pi}{2} - \arctan\left(\frac{RQ\omega}{R^2 P + \omega^2}\right), \omega \le R\sqrt{-P} \\ -\frac{\pi}{2} + \arctan\left(\frac{-RQ\omega}{R^2 P + \omega^2}\right), \omega > R\sqrt{-P}. \end{cases} \tag{16}$$

Correspondingly, the approximative transfer function $G^*_{RX_{2R}}$ and the magnitude-frequency and phase-frequency characteristics of the proposed NTD are calculated as (17)–(19), shown at the bottom of the page, respectively.

The feedforward does not change the natural frequency of NTD. According to the comparison between (16) and (19), the proposed NTD has smaller phase lags, which means that the differential estimate is more accurate.

For the specific design (9), the describing function $N(A)$ of the nonlinear function $f(z) = (\beta z)^{p/q} + z$ can be calculated as

$$\begin{aligned} N(A) &= \frac{1}{A\pi} \int_0^{2\pi} \left\{ [\beta A \sin(\omega t)]^{\frac{p}{q}} + A \sin(\omega t) \right\} \sin(\omega t) d\omega t \\ &= \frac{4}{A\pi} \int_0^{\frac{\pi}{2}} \left\{ [\beta A \sin(\omega t)]^{\frac{p}{q}} + A \sin(\omega t) \right\} \sin(\omega t) d\omega t \\ &= \frac{4}{\pi} \beta^{\frac{p}{q}} A^{\frac{p-q}{q}} \int_0^{\frac{\pi}{2}} \sin^{\frac{p+q}{q}}(\omega t) d\omega t + \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \sin^2(\omega t) d\omega t. \end{aligned} \tag{20}$$

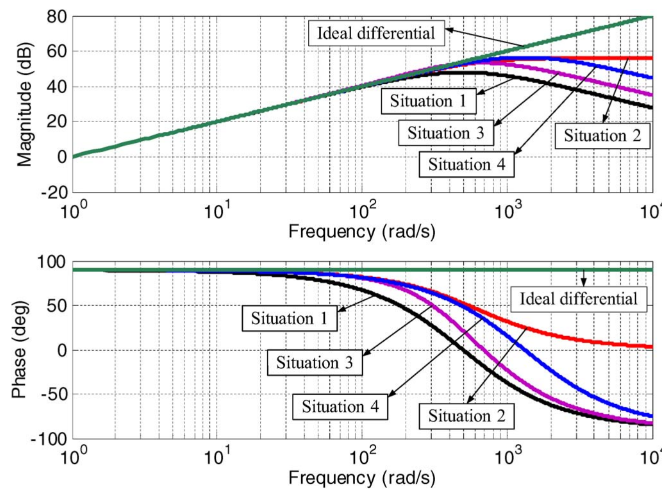

Fig. 2. Approximate analysis of the frequency characteristics. Situation 1: traditional NTD. Situation 2: proposed NTD with direct feedforward. Situation 3: proposed NTD with filter in feedforward. Situation 4: traditional NTD with greater gain.

Then, P and Q in the approximate linear system (13) of (9) are $-\alpha_1 N(A)$ and $-\alpha_2 N(A)$, respectively.

Suppose that $R = 500$, $\alpha_2 = 2\alpha_1 = 2.0$, $\beta = 30.0$, $p/q = 3$, and $\alpha = 650.0$, and an input signal of $0.001\sin(\omega t)$ is used. According to (20), there are $P = -1.02025$ and $Q = -2.0405$ in this case. Then, the approximate frequency responses are illustrated in Fig. 2 for understanding.

Situation 1 and situation 2 are, respectively, the frequency characteristics of (15) and (17), which correspond to the traditional NTD and the basic format (9) of the proposal. Obviously, the frequency characteristic of situation 2, the fundamental proposal, is closer to the ideal differential than the traditional NTD. The proposed method has smaller phase lags.

Based on the fundamental proposal, we introduce a low-pass filter with a cutoff frequency of 500 rad/s in the feedforward as Remark 3.3. The characteristic is illustrated as situation 3. The phase lags of this situation are deduced because the main components of the input signal r are still maintained within the bandwidth of the low-pass filter. Moreover, beyond the bandwidth of the filter, the characteristic of the proposed NTD trends to the traditional one, which means that the noise tolerance is strengthened compared with the fundamental proposal.

In fact, the traditional NTD also achieves the decrease in phase lags by increasing the gain R. If $R = 1330$, the differentiator has the same phase lags as the proposal in low frequency

$$G^*_{RX_{2R}}(s) = \frac{(\alpha s - R^2 P)s}{s^2 - RQs - R^2 P} \tag{17}$$

$$|G^*_{RX_{2R}}(j\omega)| = \frac{\omega\sqrt{\alpha^2\omega^2 + R^4 P^2}}{\sqrt{(\omega^2 + R^2 P)^2 + R^2 Q^2 \omega^2}} \tag{18}$$

$$\angle G^*_{RX_{2R}}(j\omega) = \begin{cases} \frac{\pi}{2} - \arctan\left(\frac{RQ\omega}{\omega^2 + R^2 P}\right) + \arctan\left(\frac{\alpha\omega}{-R^2 P}\right),\ \omega \le R\sqrt{-P} \\ -\frac{\pi}{2} + \arctan\left(\frac{-RQ\omega}{\omega^2 + R^2 P}\right) + \arctan\left(\frac{\alpha\omega}{-R^2 P}\right),\ \omega \ge R\sqrt{-P} \end{cases} \tag{19}$$

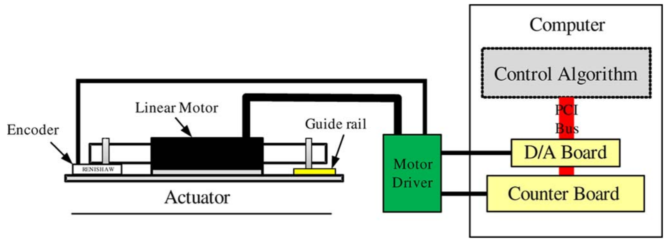

Fig. 3. Structure of the experimental servo system.

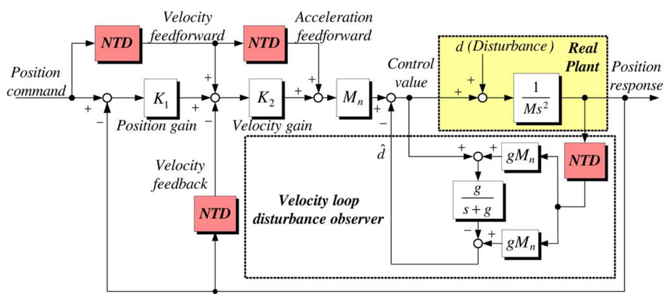

Fig. 4. Control algorithm of the servo system with differentiators.

as shown in situation 4. However, the magnitude response in high frequency is simultaneously increased. To have the same phase lags, the proposal does not need to increase gain R and has better noise tolerance than the traditional NTD.

V. Application in Motion Control

Differential calculation is widely requested in motion control systems. The differentiator is utilized to realize velocity feedback or feedforward and is also useful in disturbance rejection. This section uses the proposed differentiator in a precise servo system to improve the control performance via experiments. The system is constructed as Fig. 3, which is driven by a linear motor (GMC Hillstone S160Q). The response is detected by an optical-electricity encoder (RENISHAW RCH24Y15A30A) with a resolution of 0.1 μm.

Fig. 4 shows the control structure of the servo system. In this structure, the NTDs are employed to realize the velocity feedback, velocity feedforward, and acceleration feedforward and also to provide the velocity response for the disturbance rejection algorithm. The disturbance observer (DOB) is implemented, which is a highly robust control method used to compensate the external disturbances and model uncertainties [26]. According to the control theory, the system performs a very highly precise tracking in low frequency. Suppose that e is the servo error between the position response and the position command. In practical systems, the phase lags of the differential estimate algorithm influence the servo error e. The scalar-valued norm of the tracking error (21) is used as a measure of average tracking performance, where T_f represents the running time. In the experiments, T_f is selected as double periods of the input command

$$L_2[e] = \sqrt{\frac{1}{T_f}\int_0^{T_f} |e(\tau)|^2 d\tau}. \tag{21}$$

TABLE I
Experimental Parameters

Parameters	Marks	Value
Position control gain	K_1	10.0 s^{-2}
Velocity control gain	K_2	40.0 s^{-1}
DOB cutoff frequency	g	500.0 rad/s
Nominal mass	M_n	0.5 kg
NTD gain	R	500.0
Natural frequency parameter	α_1	1.0
Damping parameter	α_2	2.0
Feedforward gain	α	350.0
Nonlinear weighting value	β	30.0
Numerator of power parameter	p	3
Denominator of power parameter	q	1
Initial value of NTD	(x_{10}, x_{20})	(0,0)

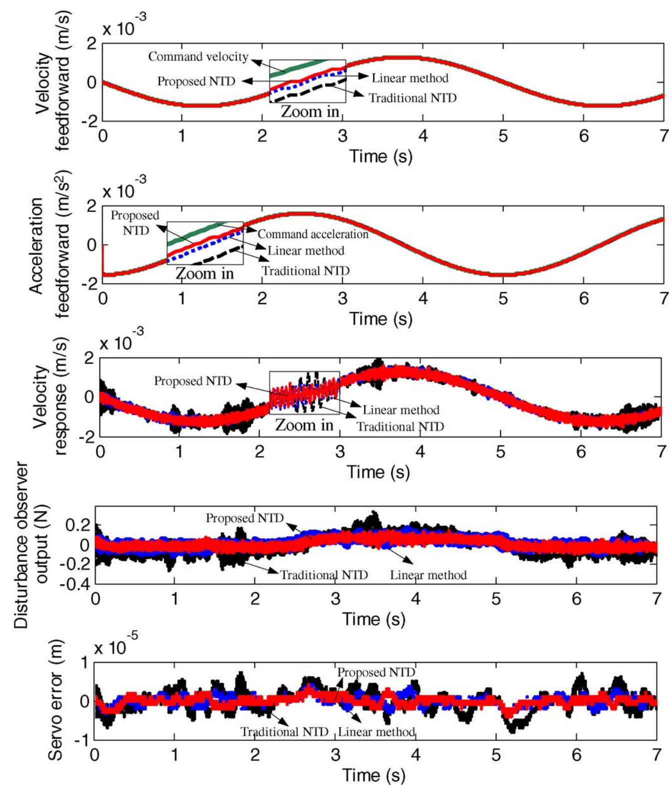

Fig. 5. Experimental results in low speed.

The required differential calculation is, respectively, realized by the proposal and the traditional NTD and also the usually used linear pseudodifferential (1) with a cutoff frequency of 500 rad/s for comparison. The parameters in the experiments are shown in Table I. The position commands are $0.001[\cos(2\pi 0.2t) - 1]$ (in meters) and $0.001[\cos(2\pi 8t) - 1]$ (in meters) in two groups of experiments. The two cases show the results in low speed and high speed, respectively. The overall algorithms are implemented in a computer by programming in Linux-RTAI. The sampling time is 0.1 ms.

Fig. 5 shows the experimental results in low speed. This figure is composed of five subfigures. According to the first two subfigures in Fig. 5, each method successfully achieves the estimations of the velocity and the acceleration. However, the proposed NTD is closer to the real value than other methods. The feedforward in the NTD effectively reduces the phase lag. Moreover, the outputs of the differentiators converge to the real value, although there are errors between the initial values and the real value according to the second subfigure.

The third subfigure in Fig. 5 illustrates the velocity response of the real servo system. All of the methods realize the differential in the presence of measurement noise. However, there is obvious oscillation in the response using the traditional NTD. This is caused by the oscillation in the disturbance compensation value of the DOB.

The output of the DOB is illustrated in the fourth subfigure. Phase lags in velocity measurement directly influence the accuracy of the disturbance estimation and compensation. Due to the phase lags caused by the traditional NTD, the observed disturbance is fluctuant. This fluctuation does not only influence the velocity response of the servo system but also degrades the servo accuracy. According to the fifth subfigure, the servo error when we use the traditional NTD is bigger than the situation of the proposed method.

In this experiment, $L_2[e]$ is 2.499×10^{-6} (in meters) using the traditional NTD, and it reduces to 1.106×10^{-6} (in meters) using the proposal. In the experimental system, a high-resolution encoder with 0.1-μm resolution is employed. Therefore, even conventional linear pseudodifferential can be used. In fact, according to the experimental results, the linear method achieves better performance than the traditional NTD. $L_2[e]$ is 1.184×10^{-6} (in meters) when the linear method is used. The most significant advantage of the traditional NTD is using double integral calculations to solve a second-order equation set to indirectly estimate the differential. The double integrals provide good noise tolerance; however, they introduce serious phase lags. As shown in (2), the conventional linear pseudodifferential only has a single integral. Therefore, the linear method has smaller phase lags than the traditional NTD when the measurement noises allow the application of the linear method.

However, the performance using the linear method is still worse than the proposed NTD because the nonlinear function makes use of the error of the signal tracking more sufficiently. According to (9), the nonlinear function provides changing gain for signal tracking. When the error between the input signal r and the output x_{1R} is greater, the speed of convergence is also greater. However, the linear method only has a fixed gain according to (2). When the input signal changes rapidly, the speed of convergence does not increase in the linear method. Moreover, in the proposal, the feedforward deduces the phase lags further. In the case of high-speed motion control, the input signal changes faster, and the tracking error may be greater. Therefore, the advantage of the nonlinear method is more obvious in this case. The proposed NTD has particularly better performance than the traditional NTD and the linear method when the system works in a high-speed situation.

In the state of high speed, the responses are illustrated in Fig. 6. The proposal achieves the best differential estimate. According to the third subfigure, the noise is hard to watch. This profits from two aspects. On the one hand, the device is a high-accuracy mechatronics system which has a big signal-to-noise ratio in the state of high speed. On the other hand, the NTD methods are noise tolerant because the algorithms acquire the differential indirectly by integral action.

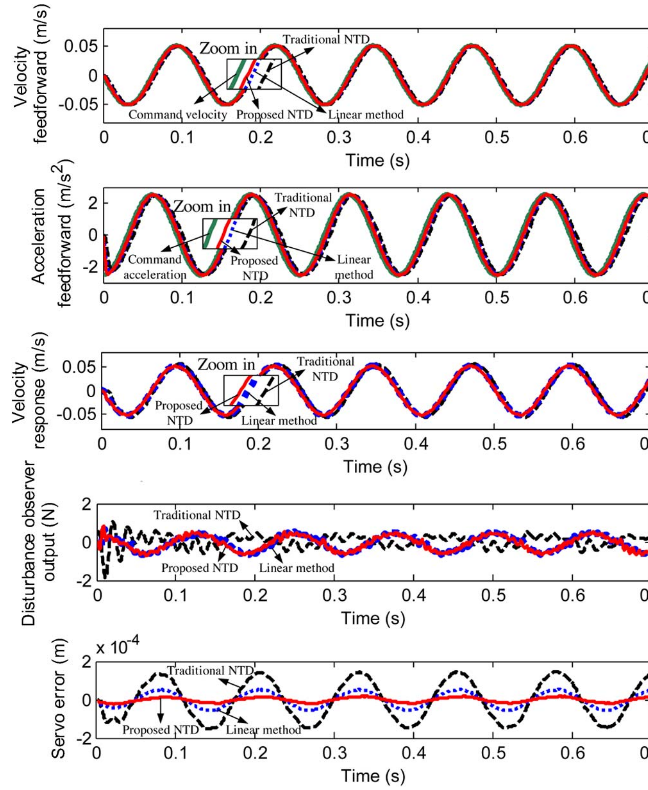

Fig. 6. Experimental results in high speed.

According to the output of the DOB, the phase lags and the oscillation in high speed are more obvious than the situation of low speed when the traditional NTD is used. Because the phase lags are greater in high speed, the influence of phase lags of the NTD is also more serious. Due to the improved differential rapidity, this oscillation of DOB is effectively weakened when the new approach is used. Correspondingly, the servo error is also decreased. $L_2[e]$ is 1.024×10^{-4} (in meters) using the traditional NTD; however, it reduces to 0.125×10^{-4} (in meters) using the proposal. The accuracy improves by 88% using the new approach. According to the results, $L_2[e]$ is 0.378×10^{-4} (in meters) when the linear pseudodifferential is used. The proposal also reduces 67% servo error compared with the linear method.

Besides the aforementioned experiments, a computer simulation is implemented to evaluate the proposal, the traditional NTD, and also the normally used linear method more fairly. The input signal is $0.001[\cos(2\pi 0.5t) - 1]$ (in meters). A noise with a magnitude of 0.5 μm is added to simulate a quantizing noise. The three methods are tuned to have the same phase of -0.24° by the least square method. The parameter $R = 500$ is kept in the proposal. To have the same phase, R should be increased to 1330 in the traditional NTD, and the cutoff frequency is increased to 710 rad/s in the traditional linear pseudodifferential.

According to Fig. 7, the linear pseudodifferential has the most serious noise in the differential estimate. The traditional NTD has smaller noise, and the proposal provides the best results with the same phase lag. The reason is that the noise is amplified by the increased gain R^2 in the traditional NTD. Moreover, the bigger cutoff frequency g allows more

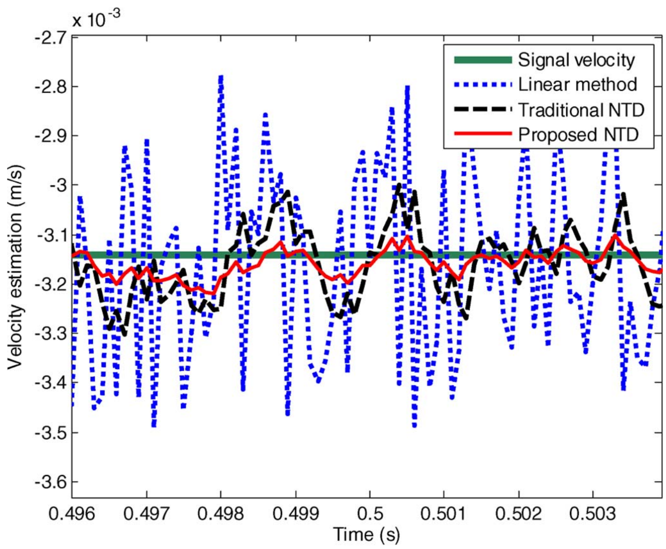

Fig. 7. Comparison of the noise sensitivity.

high-frequency noises in the linear method. Compared to this, the proposal uses a signal feedforward instead of increasing the gain to improve the phase lags, which has greater efficiency and less cost in noise tolerance. To achieve the same phase characteristics in an environment with a little greater noise, the traditional NTD indeed provides better noise tolerance than the linear method. However, the proposal is even better and more flexible than the traditional NTD.

In summary, the proposed method provides satisfactory differential estimate because of the additional freedom in the NTD design. Using the proposal, the performance of the servo control system is effectively improved in practice.

VI. Conclusion

This paper has considered the problem of the phase lags of the differentiator and has proposed an idea to improve the differential estimate by feedforward. The proposed approach extends the traditional structure and provides an additional freedom for the design of NTD. It improves the accuracy of the differential estimate compared with the traditional method. A more accurate approximative differential is achieved, which satisfies the higher requirement of engineering practice especially in the field of motion control.

The theoretical analysis is given. Moreover, the practical application in servo system is also investigated. The differentiator is employed to realize a compound control with disturbance rejection algorithm. The experimental results confirm the validity of the proposal. The proposal leads to smaller phase lags and also maintains the advantage of noise tolerance of an NTD. By applying the new approach, the accuracy of the servo system is effectively improved.

In future work, the adaptive law of the parameter R will be investigated. Although the experiments in practical system show the effectiveness, the noise in the estimated velocity response is also observable when the system moves slowly; however, it is small in the state of high speed. Therefore, fixed R is not optimal. Future research will focus on this limitation and will consider adjusting R according to the working condition.

Appendix A
Convergence of Tracking Differentiator

The convergence proof is a relatively independent problem. It is the basis of a tracking differentiator. Reference [24, Th. 2.1] illustrated the convergence of the NTD with the traditional format of the second-order differential equations set. Virtually, the Proof of Theorem 2.1 in that paper implies the convergence of the differentiator with more general format. From the process of the proof, the following corollary can be summarized, which has more extensive applicability in proposing various differentiators, such as an NTD with feedforward, NTDs with higher order or lower order equations than the traditional one. The theoretical structure of the tracking differentiator can be widely extended.

Corollary A.1: The equilibrium point **0** of such system (22) is globally asymptotically stable

$$\dot{\mathbf{z}}(t) = F\left[\mathbf{z}(t)\right],\ \mathbf{z}(0) = \mathbf{z}_0 \tag{22}$$

where

vector $\mathbf{z}(t) = [z_1(t), z_2(t), \ldots, z_n(t)]^T$ is an n-dimensional argument;

$F[\mathbf{z}(t)] = [f_1(z_1, z_2, \ldots, z_n), \ldots, f_n(z_1, z_2, \ldots, z_n)]^T$ is a functional vector, whose elements are functions of the variables in $\mathbf{z}(t)$;

$\mathbf{z}_0$ is any initial value.

$f_1, f_2, \ldots, f_n$ are local Lipschitz continuous functions. There is a smooth positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ and a continuous positive definite function $W : \mathbb{R}^n \to \mathbb{R}$ such that the following are observed: 1) when $|\mathbf{z}| \to +\infty$, $V(\mathbf{z}) \to +\ infty$; (2) $dV(\mathbf{z})/dt = \partial V/\partial z_1 \dot{z}_1 + \partial V/\partial z_2 \dot{z}_2 + \ldots + \partial V/\partial z_n \dot{z}_n \le -W(\mathbf{z})$; and (3) for any given$\sim d > 0$, $\{\mathbf{z} \in \mathbb{R}^n | V(\mathbf{z}) \le d\}$ is a bounded closed set in $\mathbb{R}^n$.

If there is a system (23) that is a perturbed system of (22)

$$\dot{\mathbf{y}}_R(t) = F\left[\mathbf{y}_R(t)\right] + \mathbf{g}_R(t), \mathbf{y}_R(0) = \mathbf{y}_{R0} \tag{23}$$

where

$\mathbf{y}_R(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T$;

$\mathbf{g}_R(t) = 1/R\mathbf{a}(t)$ is the perturbing term, $\mathbf{a}(t) = [a_1(t), a_2(t), \ldots, a_n(t)]^T$, and $\sup_{t\in[0,+\infty)} |a_i(t)|$ is bounded constants A_i $(i = 1, 2, \ldots, n)$, $\max\{A_1, A_2, \ldots, A_n\} = A$;

$\mathbf{y}_R(0)$ is nonincreasing for R;

then, there is a $\sim R_*$ such that $|\mathbf{y}_R(Rt)| < \varepsilon$ for any $\varepsilon > 0$, $a > 0$, $R > R_*$ on$\sim t \in [a, +\infty)$.

Appendix B
Transformation From (6) to (5) With Perturbation

Let $h = Rt$, and

$$\begin{cases} y_{1R}(h) = x_{1R}(t) - r(t) \\ y_{2R}(h) = \frac{x_{2R}(t)}{R}. \end{cases} \tag{24}$$

Then, $\mathbf{y}_R = (y_{1R}, y_{2R})^T$ is a solution to the system

$$\dot{\mathbf{y}}_R(t) = F\left[\mathbf{y}_R(t)\right] + \mathbf{g}_R(t) \tag{25}$$

where

$$F[\mathbf{y}_R(t)] = [y_{2R}(t), f(y_{1R}(t), y_{2R}(t))]^T \tag{26}$$

$$\mathbf{g}_R(t) = \left[-\frac{\dot{r}\left(\frac{t}{R}\right)}{R}, \frac{\alpha\dot{r}\left(\frac{t}{R}\right)}{R^2}\right]^T. \tag{27}$$

According to (26), equation set (5) can be rewritten as

$$\dot{\mathbf{z}} = F[\mathbf{z}(t)] \tag{28}$$

where $\mathbf{z} = (z_1, z_2)^T$ is a solution to system (5). Consequently, (25) is a perturbed system of (28). Then, it is reasonable that x_{1R} converges to r because $\mathbf{y}_R$ is convergent and $\mathbf{y}_R(t)$ contains the term of $x_{1R}(t) - r(t)$.

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