The topological basis expression of Heisenberg spin chain

Taotao Hu · Hang Ren · Kang Xue

Abstract In this paper, it is shown that the Heisenberg XY, XXZ, XXX, and Ising model all can be constructed from the Braid group algebra generator and the Temperley–Lieb algebra generator. And a new set of topological basis expression is presented. Through acting on the different subspaces, we get the new nontrivial six-dimensional and four-dimensional Braid group matrix representations and Temperley–Lieb matrix representations. The eigenstates of Heisenberg model can be described by the combination of the set of topological bases. It is worth mentioning that the ground state is closely related to parameter q which is the meaningful topological parameter.

Keywords Yang–Baxter equation · Knot theory · Topological basis · Quantum spin models

1 Introduction

In the integrable quantum spin systems, the one-dimensional Heisenberg model under the periodic boundary conditions is one of the fundamental models. It was originally introduced by Bethe [1] for the purpose of solving the isotropic Heisenberg spin chain, the Bethe Ansatz, and has been proven to be an invaluable tool in the field of exactly

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solved models because of its numerous refinements and generalizations. Variety of physical problems and models [2,3] have also demonstrated the versatility of this method and proved it useful. In statistical mechanics, the Temperley–Lieb algebra (TLA) first appeared as a tool to analyze various interrelated lattice models [4] and it was related to link and knot invariants [5]. Up to now, there have been many more models based on the TLA representations, such as the graph models [6–8], the RSOS models [9], and certain vertex models [10,11]. The Temperley–Lieb equivalence has also been naturally extended to the corresponding quantum counterparts of the above statistical-mechanical models, such as the quantum RSOS models and the quantum spin-S chains [10,12–15]. Recently, our team have also got that the XXX model can be constructed from the TLA generator [16]. In fact, TLA is a subalgebra of Braid group algebra (BGA). It is found that Braid group is closely linked with Yang–Baxter equation (YBE) [17,18], and the new type of braiding matrices and solutions of YBE has also been found to be related to quantum information in recent years [19–21]. In Ref. [19,22], Kauffman et al. presented a very significant result that braiding operator can be identified to the universal quantum gate (i.e. the CNOT gate). There is an earlier literature on topological quantum computation and which is all about quantum computing using braiding [23].

The topological quantum field theory (TQFT) is one of the most fantastic features of quantum theory, because it is related to quantum computing through anyons. It is shown that the 2D braid behavior under the exchange of anyons has great relation with the $\nu = 5/2$ state Fractional Quantum Hall Effect (FQHT) [24]. The topological basis plays the significant role in TQFT and it can be described in terms of graphic technique [25]. Many works have shown that topological basis has some important physical applications in topological quantum computation, quantum teleportation, and quantum entanglement [25,26]. In Ref. [25], based on the topological basis and the application of braid relation in anyon theory, authors nest TLA into 4D YBE and reduce it to 2D YBE. In $\nu = 5/2$ FQHE, quasiparticles are called Ising anyons which satisfy non-Abelian fractional statistics. The anyons obeys the fusion rules as follows,

$$\frac{1}{2} \times \frac{1}{2} = 0 + 1, \quad \frac{1}{2} \times 1 = \frac{1}{2}, \quad 1 \times 1 = 0, \tag{1}$$

$$0 \times 0 = 0, \quad 0 \times \frac{1}{2} = \frac{1}{2}, \quad 0 \times 1 = 1. \tag{2}$$

As above, there are two fusion ways for two $\frac{1}{2}$ anyons. When four $\frac{1}{2}$ anyons fuse together, we can divide the four $\frac{1}{2}$ anyons into two pairs. Both pairs either fuse 0 or to 1. According to previous theory, the well-known two orthogonal topological basis states have the form as [27–29],

$$|e_1\rangle = \frac{1}{d} \Bigg|\begin{smallmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & & & \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{smallmatrix}\Bigg| = \frac{1}{d} \underset{1}{\cup}\underset{2\ 3}{\cup}\underset{4}{},$$

$$|e_2\rangle = \frac{1}{\sqrt{d^2-1}} \Bigg|\begin{smallmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & & & \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \end{smallmatrix}\Bigg| = \frac{1}{\sqrt{d^2-1}}(\underset{1}{\cup}\underset{2\ 3\ 4}{\cup} - \underset{1}{\cup}\underset{2\ 3}{\cup}\underset{4}{\cup}). \tag{3}$$

where the parameter d denotes the value of a unknotted loop $\bigcirc$. In the middle fusion chains, the internal edges obey the fusion rules at each trivalent vertex. On the right-hand sides, from the conformal basis to the Kauffman graph, Jones–Wenzl projector operators have been applied, i.e.

$$\Pi_0 = \frac{1}{d} \overset{i \quad j}{\underset{\cap}{\cup}}, \quad \Pi_1 = \left|\ \right| - \frac{1}{d} \overset{i \quad j}{\underset{\cap}{\cup}}. \tag{4}$$

These just indicate the importance of braid group in quantum information and topological theory. In this paper, we investigate the non-standard solution of braid group matrix representation which is independent of Temperley–Lieb matrix representation. So we combine the TLA and BGA in order to obtain more Heisenberg model, the combination of these two algebra is just the Birman–Wenzl Algebra (BWA) [30].

Our aim in this work is to connect new topological basis states with more Heisenberg spin chain, and we use a graphic method to construct the exact solutions of the four-qubit Heisenberg spin chain, and study some properties of the topological basis states of these system. This paper is organized as follows: in the second section, we recall the BGA, TLA and construct a set of complete orthonormal bases of four-qubit spin chain with topological bases. Through acting on the subspaces, we get the new nontrivial six-dimensional (6D) and four-dimensional Braid group matrix representations and Temperley–Lieb matrix representations, which all satisfy the reduced TLA and BGA relation respectively. it shows that the Hamiltonian of Heisenberg XY, XXZ, XXX, and Ising model all can be constructed from the BGA generator and the TLA generators. Then we show a graphic method of constructing the exact solutions for a closed four-qubit Heisenberg XY and Ising spin chain. The eigenstates of XY and Ising model can be expressed by topological bases, and the ground state is closely related to the parameter q which is the meaningful topological parameter.

2 BGA, TLA, and a new expression of topological basis

We first briefly review the theory of the TLA [31]. For each natural number m, the TLA $TL_m(d)$ is generated by $\{I, U_1, U_2 \ldots U_{m-1}\}$ with the TLA relations:

$$\begin{cases} U_i^2 = dU_i & 1 \leq i \leq m - 1 \\ U_i U_{i\pm 1} U_i = U_i & 1 \leq i \leq m \\ U_i U_j = U_j U_i & |i - j| \geq 2 \end{cases} \tag{5}$$

where d is the unknotted loop $\bigcirc$ in the knot theory which does not depend on the sites of the lattices. The notation $U_i \equiv U_{i,i+1}$ is used. The U_i represents $1_1 \otimes 1_2 \otimes 1_3 \otimes \cdots \otimes 1_{i-1} \otimes U \otimes 1_{i+2} \cdots 1_m$, and 1_j represents the unit matrix in the jth space V_j. In addition, the TLA is easily understood in terms of knot diagrams in Ref. [32]. According to Kauffman's graphs, it can be expressed as,

$$
\begin{cases}
U_i \longrightarrow \;\overset{i\;\;i+1}{\cup\cap}\;, \quad U_i^2 = dU_i \longrightarrow \;\overset{i\;\;i+1}{\underset{\cap}{\cup}}\; = \bigcirc\,\overset{i\;\;i+1}{\cup\cap} \\[10pt]
U_i U_{i+1} U_i = U_i \longrightarrow \;\overset{i\;i+1\,i+2}{\cdots}\; = \;\overset{i\;i+1\,i+2}{\cdots} \\[10pt]
U_i U_j = U_j U_i \longrightarrow \;\overset{i\;i+1\,j\;j+1}{\cdots}\; = \;\overset{i\;i+1\,j\;j+1}{\cdots}
\end{cases}
\tag{6}
$$

Then, we review the theory of braid groups to keep the paper self-contained. Let S_n denotes the braid group on n strands [17]. S_n is generated by elementary braids $\{S_1, S_2, \ldots, S_{n-1}\}$ with the braid relations,

$$
\begin{cases}
S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1} & 1 \le i < n - 2 \\
S_i S_j = S_j S_i & |i - j| \ge 2
\end{cases}
\tag{7}
$$

where the notation $S_i \equiv S_{i,i+1}$ is used, $S_{i,i+1}$ represents $1_1 \otimes 1_2 \otimes 1_3 \otimes \cdots \otimes 1_{i-1} \otimes S \otimes 1_{i+2} \cdots 1_m$, and 1_j is the unit matrix of the j-th particle. Also using Kauffman's graphs [32], it can be expressed as

$$
\begin{cases}
S_i \longrightarrow \;\overset{i\;\;i+1}{\times}\;, \quad S_i S_i^- = I \longrightarrow \;\overset{i\;i+1\,i+2}{\cdots}\; = \;\overset{i\;i+1\,i+2}{|||} \\[10pt]
S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1} \longrightarrow \;\overset{i\;i+1\,i+2}{\cdots}\; = \;\overset{i+1\,i+2}{\cdots} \\[10pt]
S_i S_j = S_j S_i \longrightarrow \;\overset{i\;i+1\,j\;\;j+1}{\cdots}\; = \;\overset{i\;i+1\,j\;\;j+1}{\cdots}
\end{cases}
\tag{8}
$$

According to Kauffman [19], there is the decomposition for the standard solution of braid groups matrix representation:

$$
S_i = \;\overset{i\;\;i+1}{\times}\; = \alpha \;\overset{i\;\;i+1}{||}\; + \alpha^- \;\overset{i\;\;i+1}{\cup\cap}
\tag{9}
$$

where $-(\alpha^2 + \alpha^{-2}) = d$, Eq. (9) just indicate that the standard solution of braid groups matrix representation can be expressed by TL matrix representation U. The Hermitian matrix U with $d = q + q^{-1}$, which satisfies the TLA relations Eq. (5) and can be used to construct the well-known six-vertex model [33], has the representation,

$$
U = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & q & \eta & 0 \\
0 & \eta^{-1} & q^{-1} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\tag{10}
$$

where $\eta = e^{i\varphi}$; in this paper, we consider the case of $\varphi = \pi$ and $\eta = -1$ to construct Heisenberg model. Corresponding to Eq. (10), the typical standard solution of braid groups matrix representation S for the six-vortex models, which satisfies the BGA relations Eq. (7), takes the representation,

$$S = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & -\eta & 0 \\ 0 & -\eta^{-1} & q - q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{11}$$

Observing Eqs. (10) and (11), it is consistent with Eq. (9) that the typical standard solution S can be expressed in terms of TL matrix U as

$$S = \rho(I + fU) \tag{12}$$

where $\rho = q$, $f = -q^{-1}$. For the six-vortex models, there is another unique braid matrix representation [19,34] which is the non-standard solution structure,

$$S' = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & -\eta & 0 \\ 0 & -\eta^{-1} & q - q^{-1} & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}, \tag{13}$$

Observing Eqs. (10) and (13), it is easy to find that the unique non-standard braid solution S' cannot be expressed in terms of Temperley–Lieb (T–L) matrix U. It means that the non-standard braid solution S' is independent of T–L matrix representation U. Based on this, we can combine the BGA generator S' and the TLA generator U to construct more Heisenberg model in this paper.

Here we introduce four graphs and their spin realization as follows:

$$\frac{1}{\sqrt{d}} \bigcup{}^{i\ j} = |\psi_d\rangle_{ij} = \frac{1}{\sqrt{d}} \left[q^{\frac{1}{2}} |\uparrow\downarrow\rangle_{ij} - q^{-\frac{1}{2}} |\downarrow\uparrow\rangle_{ij} \right] = {}_{ij}\langle\psi_d| \big]^{\dagger} = \frac{1}{\sqrt{d}} \left[\bigcap{}_{i\ j} \right]^{\dagger},$$

$$\frac{1}{\sqrt{d}} \bigcup{}^{i\ j} = |\psi_0\rangle_{ij} = \frac{1}{\sqrt{d}} \left[q^{\frac{1}{2}} |\uparrow\downarrow\rangle_{ij} + q^{-\frac{1}{2}} |\downarrow\uparrow\rangle_{ij} \right] = {}_{ij}\langle\psi_0| \big]^{\dagger} = \frac{1}{\sqrt{d}} \left[\bigcap{}_{i\ j} \right]^{\dagger},$$

$$\frac{1}{\sqrt{d}} \bigcup{}^{i\ j} = |\xi_0\rangle_{ij} = |\uparrow\uparrow\rangle_{ij} = {}_{ij}\langle\xi_0| \big]^{\dagger} = \frac{1}{\sqrt{d}} \left[\bigcap{}_{i\ j} \right]^{\dagger},$$

$$\frac{1}{\sqrt{d}} \bigcup{}^{i\ j} = |\eta_0\rangle_{ij} = |\downarrow\downarrow\rangle_{ij} = {}_{ij}\langle\eta_0| \big]^{\dagger} = \frac{1}{\sqrt{d}} \left[\bigcap{}_{i\ j} \right]^{\dagger}.$$

where the solid line, the dashed line, the dotted line, and the dash-dotted line are used respectively, and the notation $\uparrow$ ($\downarrow$) denotes spin up (down), and the nota-

tion $|\alpha\beta\rangle_{ij}$ is the abbreviated form of $|\alpha\rangle_i \otimes |\beta\rangle_j$. Then, we get a set of complete orthonormal topological bases which are combined by these four graphs as follows:

$$
|e_1\rangle = \frac{1}{d}\ \ ;\quad |e_2\rangle = \frac{1}{d}\ \ ;\quad |e_3\rangle = \frac{1}{d}\ \ ;\quad |e_3\rangle = \frac{1}{d}\ \ ,
$$
$$
|e_5\rangle = \frac{1}{d}\ \ ;\quad |e_6\rangle = \frac{1}{d}\ \ ;\quad |e_7\rangle = \frac{1}{d}\ \ ;\quad |e_8\rangle = \frac{1}{d}\ \ ,
$$
$$
|e_9\rangle = \frac{1}{d}\ \ ;\quad |e_{10}\rangle = \frac{1}{d}\ \ ;\quad |e_{11}\rangle = \frac{1}{d}\ \ ;\quad |e_{12}\rangle = \frac{1}{d}\ \ ,
$$
$$
|e_{13}\rangle = \frac{1}{d}\ \ ;\quad |e_{14}\rangle = \frac{1}{d}\ \ ;\quad |e_{15}\rangle = \frac{1}{d}\ \ ;\quad |e_{16}\rangle = \frac{1}{d}\ \ .
$$

$$(14)$$

Acting the operator S' of Eq. (13) and U of Eq. (10) on topological bases of Eq. (14) respectively, we can get a set of new braid matrix and T–L matrix representations. We get four subspaces and the first subspace is spanned by $\{|e_1\rangle, |e_2\rangle, |e_3\rangle, |e_4\rangle, |e_5\rangle, |e_6\rangle\}$. When the operator S' and U, respectively, act on the first subspace, the 6D braid and T–L matrix representations are

$$
S_A^{(1)} = \begin{pmatrix}
-q^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & -q^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & -q^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & q & 0 & 0 \\
0 & 0 & 0 & 0 & q & 0 \\
0 & 0 & 0 & 0 & 0 & q
\end{pmatrix},\quad
S_B^{(1)} = \begin{pmatrix}
\frac{q^2-1}{d} & \frac{q}{d} & \frac{1}{dq} & \frac{q}{d} & -\frac{1}{dq} & 0 \\
\frac{q}{d} & 0 & -\frac{1}{d} & \frac{1}{d} & 0 & \frac{1}{dq} \\
\frac{1}{dq} & -\frac{1}{d} & q-q^{-1} & 0 & -\frac{1}{d} & -\frac{q}{d} \\
\frac{q}{d} & \frac{1}{d} & 0 & 0 & \frac{1}{d} & -\frac{1}{dq} \\
-\frac{1}{dq} & 0 & -\frac{1}{d} & \frac{1}{d} & q-q^{-1} & -\frac{q}{d} \\
0 & \frac{1}{dq} & -\frac{q}{d} & -\frac{1}{dq} & -\frac{q}{d} & \frac{1-q^{-2}}{d}
\end{pmatrix},
$$

$$
S_C^{(1)} = \begin{pmatrix}
-q^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & q & 0 & 0 & 0 & 0 \\
0 & 0 & q & 0 & 0 & 0 \\
0 & 0 & 0 & -q^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & -q^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & q
\end{pmatrix},\quad
S_D^{(1)} = \begin{pmatrix}
\frac{q^2-1}{d} & -\frac{1}{dq} & \frac{q}{d} & \frac{1}{dq} & \frac{q}{d} & 0 \\
-\frac{1}{dq} & q-q^{-1} & \frac{1}{d} & -\frac{1}{d} & 0 & -\frac{q}{d} \\
\frac{q}{d} & \frac{1}{d} & 0 & 0 & \frac{1}{d} & -\frac{1}{dq} \\
\frac{1}{dq} & -\frac{1}{d} & 0 & q-q^{-1} & -\frac{1}{d} & -\frac{q}{d} \\
\frac{q}{d} & 0 & \frac{1}{d} & -\frac{1}{d} & 0 & \frac{1}{dq} \\
0 & -\frac{q}{d} & -\frac{1}{dq} & -\frac{q}{d} & \frac{1}{dq} & \frac{1-q^{-2}}{d}
\end{pmatrix},
$$

$$(15)$$

and

$$U_A^{(1)} = \begin{pmatrix} d & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad U_B^{(1)} = \begin{pmatrix} \frac{1}{d} & 0 & -\frac{1}{dq} & -\frac{q}{d} & 0 & -\frac{1}{d} \\ 0 & \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & \frac{q-q^{-1}}{d} \\ -\frac{1}{dq} & \frac{1}{d} & q^{-1} & 0 & \frac{1}{d} & \frac{q}{d} \\ -\frac{q}{d} & -\frac{1}{d} & 0 & q & -\frac{1}{d} & \frac{1}{dq} \\ 0 & \frac{1}{d} & \frac{1}{d} & -\frac{1}{d} & \frac{1}{d} & \frac{q-q^{-1}}{d} \\ -\frac{1}{d} & \frac{q-q^{-1}}{d} & \frac{q}{d} & \frac{1}{dq} & \frac{q-q^{-1}}{d} & \frac{d^2-3}{d} \end{pmatrix},$$

$$U_C^{(1)} = \begin{pmatrix} d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad U_D^{(1)} = \begin{pmatrix} \frac{1}{d} & 0 & -\frac{q}{d} & -\frac{1}{dq} & 0 & -\frac{1}{d} \\ 0 & \frac{1}{d} & -\frac{1}{d} & \frac{1}{d} & \frac{1}{d} & \frac{q-q^{-1}}{d} \\ -\frac{q}{d} & -\frac{1}{d} & q & 0 & -\frac{1}{d} & \frac{1}{dq} \\ -\frac{1}{dq} & \frac{1}{d} & 0 & q^{-1} & \frac{1}{d} & \frac{q}{d} \\ 0 & \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & \frac{q-q^{-1}}{d} \\ -\frac{1}{d} & \frac{q-q^{-1}}{d} & \frac{1}{d} & \frac{q}{d} & \frac{q-q^{-1}}{d} & \frac{d^2-3}{d} \end{pmatrix},$$

$$(16)$$

where $(S_A)_{ij} = \langle e_i|S_{12}|e_i\rangle$, $(S_B)_{ij} = \langle e_i|S_{23}|e_i\rangle$, $(S_C)_{ij} = \langle e_i|S_{34}|e_i\rangle$, $(S_D)_{ij} = \langle e_i|S_{41}|e_i\rangle$, so do the matrix U. It is worth mentioning that these representations in Eqs. (15) and (16) are the new 6D braid and Temperley–Lieb matrix representations, and they all satisfy the 6D TLA and 6D BGA relation respectively. The 6D braid matrix representations $S_A^{(1)} \neq S_C^{(1)}$, $S_B^{(1)} \neq S_D^{(1)}$ and the Temperley–Lieb matrix representations $U_A^{(1)} \neq U_C^{(1)}$, $U_B^{(1)} \neq U_D^{(1)}$ indicate that there is not the symmetry of exchanging pair indices $12 \leftrightarrow 34$ and $23 \leftrightarrow 41$ for the first subspace.

The second and the third subspaces are spanned by $\{|e_7\rangle, |e_8\rangle, |e_9\rangle, |e_{10}\rangle\}$, and $\{|e_{11}\rangle, |e_{12}\rangle, |e_{13}\rangle, |e_{14}\rangle\}$, respectively. When the operator S' and U, respectively, act on the second and the third subspaces, the 4D braid and T-L matrix representations are

$$S_A^{(2)} = \begin{pmatrix} -q^{-1} & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad S_B^{(2)} = \frac{1}{d}\begin{pmatrix} q^2 & -1 & -q^{-1} & q \\ -1 & q^2 & -q^{-1} & q \\ -q^{-1} & -q^{-1} & dq-q^{-2} & 1 \\ q & q & 1 & 1 \end{pmatrix},$$

$$S_C^{(2)} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & -q^{-1} & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad S_D^{(2)} = \begin{pmatrix} q^2 & -1 & q & -q^{-1} \\ -1 & q^2 & q & -q^{-1} \\ q & q & 1 & 1 \\ -q^{-1} & -q^{-1} & 1 & dq-q^{-2} \end{pmatrix}, \quad (17)$$

and

$$U_A^{(2)} = U_A^{(3)} = \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad U_B^{(2)} = U_D^{(3)} = \frac{1}{d} \begin{pmatrix} 1 & 1 & q^{-1} & -q \\ 1 & 1 & q^{-1} & -q \\ q^{-1} & q^{-2} & q^{-2} & -1 \\ -q & -q & -1 & q^2 \end{pmatrix},$$

$$U_C^{(2)} = U_C^{(3)} = \begin{pmatrix} -q^{-1} & 0 & 0 & 0 \\ 0 & -q^{-1} & 0 & 0 \\ 0 & 0 & -q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad U_D^{(2)} = U_B^{(3)} = \frac{1}{d} \begin{pmatrix} 1 & 1 & -q & q^{-1} \\ 1 & 1 & -q & q^{-1} \\ -q & -q & q^2 & -1 \\ q^{-1} & q^{-1} & -1 & q^{-2} \end{pmatrix},$$

$$(18)$$

and

$$S_A^{(3)} = \begin{pmatrix} -q^{-1} & 0 & 0 & 0 \\ 0 & -q^{-1} & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}, \quad S_B^{(3)} = \frac{1}{d} \begin{pmatrix} -1 & -1 & -q^{-1} & -q^{-1} \\ -1 & q^{-2} - dq & q & q \\ -q^{-1} & q & -q^{-2} & 1 \\ -q^{-1} & q & 1 & -q^{-2} \end{pmatrix},$$

$$S_C^{(3)} = \begin{pmatrix} -q^{-1} & 0 & 0 & 0 \\ 0 & -q^{-1} & 0 & 0 \\ 0 & 0 & -q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad S_D^{(3)} = \frac{1}{d} \begin{pmatrix} q^{-2} - dq & -1 & q & q \\ -1 & -1 & -q^{-1} & -q^{-1} \\ q & -q^{-1} & -q^{-2} & 1 \\ q & -q^{-1} & 1 & -q^{-2} \end{pmatrix},$$

$$(19)$$

Above all the new 4D braid and 4D Temperley–Lieb satisfy the 4D BGA and 4D TLA relation, respectively. It is same as the first subspace, $S_A^{(2/3)} \neq S_C^{(2/3)}$, $S_B^{(2/3)} \neq S_D^{(2/3)}$ and $U_A^{(2/3)} \neq U_C^{(2/3)}$, $U_B^{(2/3)} \neq U_D^{(2/3)}$. It also does not satisfy the exchange symmetry. The last subspaces are spanned by $\{|e_{15}\rangle, |e_{16}\rangle\}$

$$S_A^{(4)} = S_B^{(4)} = S_C^{(4)} = S_D^{(4)} = \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix}, \tag{20}$$

and

$$U_A^{(4)} = U_B^{(4)} = U_C^{(4)} = U_D^{(4)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{21}$$

where all the 2D braid and 2D Temperley–Lieb satisfy the 2D BGA and 2D TLA relation, respectively.

3 The graphic solutions and the topological basis

In this section, we present that the Hamiltonian of Heisenberg XY, XXZ, XXX, and Ising model all can be constructed from the BGA generator and the TLA generators, and we show a graphic method of constructing the exact solutions for a closed four-qubit Heisenberg XY and Ising spin chain.

In the Eqs. (13) and (10), we consider the case of $\eta = -1$. For the i-th and $(i+1)$-th lattices, the matrix S' and U can be denoted in terms of local spin operators,

$$
\begin{aligned}
S'_{i,i+1} &= s_i^+ s_{i+1}^- + s_i^- s_{i+1}^+ + q^{-1} s_i^3 + q s_{i+1}^3 + \frac{1}{2}(q - q^{-1}) I \\
&= 2 H_{i,i+1}^{(1)} + \frac{1}{2}(q - q^{-1}) I,
\end{aligned} \tag{22}
$$

and

$$
U_{i,i+1} = -2 \left[\frac{1}{2}(s_i^+ s_{i+1}^- + s_i^- s_{i+1}^+) + \frac{1}{2}(q + q^{-1}) s_i^3 s_{i+1}^3 \right] + \frac{1}{4}(q + q^{-1}) I \tag{23}
$$

It is worth mentioning that the Hamiltonian $H_{i,i+1}^{(1)}$ in Eq. (22) is just the Heisenberg XY model. So the XY model can be constructed from the Braid generator as follows:

$$
H_{i,i+1} = \frac{1}{2} \left[S'_{i,i+1} - \frac{1}{2}(q - q^{-1}) I \right] \tag{24}
$$

In Sect. 2, it is shown that the braid matrix representation S' is independent of the T–L matrix representation U, so we can combine these two matrix representation as follows:

$$
\begin{aligned}
S'_{i,i+1} &+ \alpha U_{i,i+1} \\
&= (1 - \alpha)(s_i^+ s_{i+1}^- + s_i^- s_{i+1}^+) - \alpha(q + q^{-1}) s_i^3 s_{i+1}^3 \\
&\quad + \left(q^{-1} + \frac{\alpha(q - q^{-1})}{2} \right) s_i^3 + \left(q - \frac{\alpha(q - q^{-1})}{2} \right) s_{i+1}^3 + A I \\
&= H_{i,i+1}^{(2)} + A I,
\end{aligned} \tag{25}
$$

where $A = ((2 + \alpha)q + (\alpha - 2)q^{-1})/4$. According to Eq. (25), it is easy to see that the Hamiltonian $H_{i,i+1}^{(2)}$ in Eq. (25) is just the Heisenberg XXX model when $\alpha = \frac{2}{2-d}$, the Heisenberg XXZ model when $\alpha \neq \frac{2}{2-d}$, and the Heisenberg Ising model when $\alpha = 1$, respectively. So we construct the Heisenberg XY, XXZ, XXX, and Ising model via the BGA generator and the TLA generator. Then, we will mainly show a graphic method of constructing the exact solutions for a closed four-qubit Heisenberg XY spin chain and Ising spin chain, we also investigate the particular properties of the topological basis states in these systems.

In the following, we discuss the Hamiltonian of a closed four-qubit Heisenberg spin chain under the periodic boundary conditions given by

$$H = J \sum_{i=1}^{4} H_{i,i+1}. \tag{26}$$

where J is the real coupling coefficient. The coupling constant $J > 0$ corresponds to the antiferromagnetic case, and $J > 0$ corresponds to the ferromagnetic case. For the four-qubit Heisenberg XY spin chain, according to Eq. (24), it is easy to check that the eigenstates of the XY Hamiltonian are the same as the eigenstates of $\Gamma = \sum_{i=1}^{4} S'_{i,i+1}$. So via the combinations of the above topological basis states $\{|e_1\rangle, i = 1, 2, \ldots, 16\}$, one can construct the exact solutions for the XY Hamiltonian as follows:

$$|\Phi_1\rangle = -\frac{\alpha}{2dq}|e_1\rangle + \frac{\beta(q-q^{-1})}{4\alpha d}(|e_2\rangle + |e_5\rangle) + \frac{\beta}{4\alpha}(|e_3\rangle + |e_4\rangle) + \frac{q(d^2-2)}{2\alpha d}|e_6\rangle,$$

$$|\Phi_2\rangle = \frac{q-q^{-1}}{\sqrt{2d^2+8}}(|e_1\rangle + |e_6\rangle)),$$

$$|\Phi_3\rangle = \frac{\sqrt{2}}{2}(-|e_2\rangle + |e_5\rangle)),$$

$$|\Phi_4\rangle = -\frac{d}{\sqrt{2d^2-4}}|e_2\rangle + \frac{q-q^{-1}}{\sqrt{2d^2-4}}|e_4\rangle,$$

$$|\Phi_5\rangle = -\frac{d}{\sqrt{2d^2-4}}|e_2\rangle + \frac{q-q^{-1}}{\sqrt{2d^2-4}}|e_3\rangle,$$

$$|\Phi_6\rangle = -\frac{\gamma}{2d}|e_1\rangle - \frac{\mu(q-q^{-1})}{4\gamma d}(|e_2\rangle + |e_5\rangle) + \frac{\mu}{4\gamma}(|e_3\rangle + |e_4\rangle) + \frac{q(d^2-2)}{2\gamma d}|e_6\rangle,$$

$$|\Phi_7\rangle = \frac{\sqrt{2}}{2}(-|e_7\rangle + |e_8\rangle)),$$

$$|\Phi_8\rangle = \frac{\sqrt{2}}{2}(-|e_9\rangle + |e_{10}\rangle)),$$

$$|\Phi_9\rangle = -\frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2\sqrt{d}}|e_7\rangle - \frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2\sqrt{d}}|e_8\rangle + \frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{2\sqrt{d}}|e_9\rangle - \frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{2\sqrt{d}}|e_{10}\rangle, \tag{27}$$

$$|\Phi_{10}\rangle = \frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{2\sqrt{d}}|e_7\rangle + \frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{2\sqrt{d}}|e_8\rangle + \frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2\sqrt{d}}|e_9\rangle + \frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2\sqrt{d}}|e_{10}\rangle,$$

$$|\Phi_{11}\rangle = \frac{\sqrt{2}}{2}(-|e_{11}\rangle + |e_{12}\rangle)),$$

$$|\Phi_{12}\rangle = \frac{\sqrt{2}}{2}(-|e_{13}\rangle + |e_{14}\rangle)),$$

$$|\Phi_{13}\rangle = -\frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2\sqrt{d}}|e_{11}\rangle - \frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2\sqrt{d}}|e_{12}\rangle + \frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{2\sqrt{d}}|e_{13}\rangle - \frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{2\sqrt{d}}|e_{14}\rangle,$$

$$|\Phi_{14}\rangle = \frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{2\sqrt{d}}|e_{11}\rangle + \frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{2\sqrt{d}}|e_{12}\rangle + \frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2\sqrt{d}}|e_{13}\rangle + \frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2\sqrt{d}}|e_{14}\rangle,$$

$$|\Phi_{15}\rangle = |e_{15}\rangle,$$

$$|\Phi_{16}\rangle = |e_{16}\rangle.$$

Here we have set $\alpha = \sqrt{1 + 2q\beta + q^4}$, $\beta = \sqrt{2}+2q+\sqrt{2}q^2$, $\gamma = \sqrt{1 - 2q\mu + q^4}$, $\mu = \sqrt{2} - 2q + \sqrt{2}q^2$,
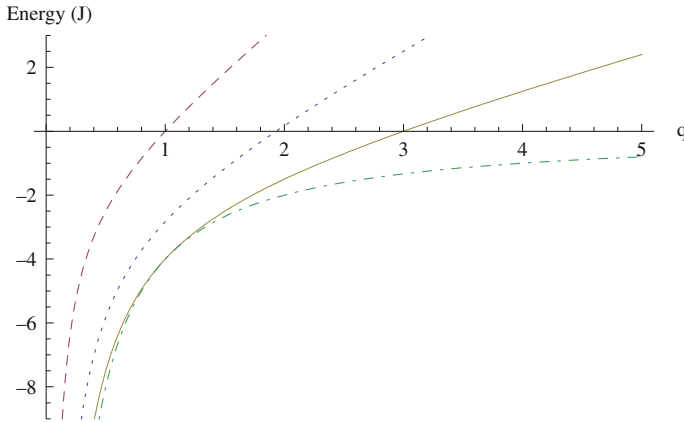
Energy (J)

Fig. 1 The *thick line* is $E_{13} = J(q - 3q^{-1} - 2)$. The *dashed line* is $E_9 = J(3q - q^{-1} - 2)$. The *dotted line* is $E_1 = 2J(q - q^{-1} - \sqrt{2})$. The *dot dashed line* is $E_{16} = -4Jq^{-1}$

The corresponding eigenvalues are

$$E_2 = E_3 = E_4 = E_5 = 2J(q - q^{-1}), \quad E_6 = 2J(q - q^{-1} + \sqrt{2}),$$
$$E_{14} = J(q - 3q^{-1} + 2), \quad E_{15} = 4Jq, \quad E_{10} = J(3q - q^{-1} + 2),$$
$$E_{11} = E_{12} = J(q - 3q^{-1}), \quad E_7 = E_8 = J(3q - q^{-1}), \quad E_1 = 2J(q - q^{-1} - \sqrt{2}),$$
$$E_9 = J(3q - q^{-1} - 2), \quad E_{13} = J(q - 3q^{-1} - 2), \quad E_{16} = -4Jq^{-1}.$$

The eigenenergy of the XY Hamiltonian is shown in Fig. 1. It shows the condition of ground energy with the value of parameter q changing. Here we consider the antiferromagnetic case, $J > 0$. The expression of $\sqrt{q}$ in Eq. (27) implies that the parameter q must be a nonegative number. So according to Fig. 1, we can get the conclusion that the ground state energy of the XY Hamiltonian is $E_{16} = -4Jq^{-1}$. There is energy degeneracy at the point of $q = 1$, accordingly $E_{13} = J(q - 3q^{-1} - 2) = E_{16} = -4Jq^{-1}$. It means that the ground state is closely related to the parameter q which is the meaningful topological parameter and is very significance in physics models [20,21,35].

For the four-qubit Heisenberg Ising spin chain, according to Eq. (25), it is easy to check that the eigenstates of the Ising Hamiltonian are the same as the eigenstates of $\Gamma = \sum_{i=1}^{4}(S'_{i,i+1} + U_{i,i+1})$. So via the combinations of the above topological basis states $\{|e_1\rangle, i = 1, 2, \ldots, 16\}$, one can also construct the exact solutions for the Ising Hamiltonian as follows:

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}(-|e_1\rangle + |e_6\rangle), \quad |\Phi_4\rangle = \frac{1}{\sqrt{2}}(-|e_2\rangle + |e_5\rangle),$$
$$|\Phi_2\rangle = \frac{1}{\sqrt{d}}\left(\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)|e_1\rangle + |e_2\rangle + |e_5\rangle\right),$$
$$|\Phi_3\rangle = \frac{1}{\sqrt{d}}\left(-\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)|e_2\rangle + |e_1\rangle + |e_6\rangle\right), \tag{28}$$
$$|\Phi_5\rangle = |e_4\rangle, \quad |\Phi_6\rangle = |e_3\rangle, \quad |\Phi_7\rangle = |e_{10}\rangle, \quad |\Phi_8\rangle = |e_9\rangle,$$
$$|\Phi_9\rangle = |e_8\rangle, \quad |\Phi_{10}\rangle = |e_7\rangle, \quad |\Phi_{11}\rangle = |e_{14}\rangle, \quad |\Phi_{12}\rangle = |e_{13}\rangle,$$
$$|\Phi_{13}\rangle = |e_{12}\rangle, \quad |\Phi_{14}\rangle = |e_{11}\rangle, \quad |\Phi_{15}\rangle = |e_{15}\rangle, \quad |\Phi_{16}\rangle = |e_{16}\rangle.$$
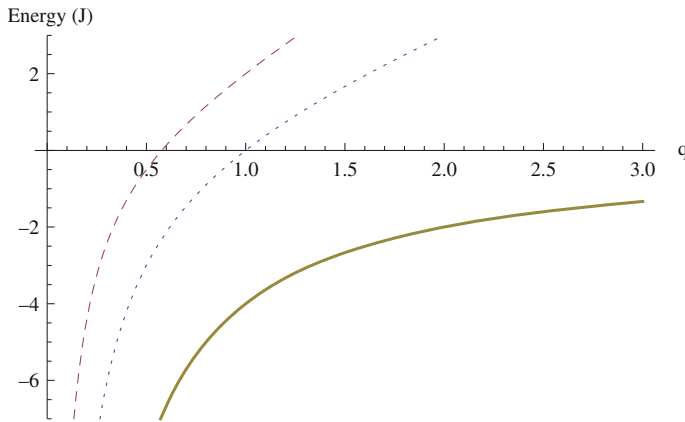
Fig. 2 The *thick line* is $E_{16} = -4Jq^{-1}$, The *dashed line* is $E_3 = J(3q - q^{-1})$, The *dotted line* is $E_{11} = 2J(q - q^{-1})$

The corresponding eigenvalues are

$$E_1 = E_2 = E_7 = E_8 = E_9 = E_{10} = E_{15} = 4Jq, \quad E_{16} = -4Jq^{-1},$$
$$E_3 = E_4 = E_5 = E_6 = J(3q - q^{-1}), \quad E_{11} = E_{12} = E_{13} = E_{14} = 2J(q - q^{-1}).$$

The eigenenergy of the Ising Hamiltonian are shown in Fig. 2. It shows the condition of ground energy with the value of parameter q changing. Also we consider the antiferromagnetic case, $J > 0$. The expression of $\sqrt{q}$ in Eq. (28) also implies that the parameter q must be a nonegative number. So according to Fig. 2, we can get the conclusion that the ground state energy of the Ising Hamiltonian is also $E_{16} = -4Jq^{-1}$. But there is not energy degeneracy for the Ising Hamiltonian. It is worth noting that the ground state of the closed four-qubit Heisenberg XY spin chain and the Ising spin chain all falls on the topological basis states $|e_{16}\rangle$. It also means that the ground state is closely related to the topological parameter q which is very significance in physics models [20,21,35].

4 Summary

In conclusion, via constructing the Heisenberg XY, XXZ, XXX, and Ising model from the BGA generator and the TLA generator, we have connected the topological basis states with more Heisenberg spin chain. We present a new set of topological basis expression. Through acting on the different subspaces, we get the new nontrivial 6D and 4D Braid group matrix representations and Temperley–Lieb matrix representations which all satisfy the reduced BGA and TLA relations. We mainly show a graphic method of constructing the exact solutions for a four-qubit Heisenberg XY spin chain and Ising spin chain, and the eigenstates of XY and Ising model can be described by the combination of the set of topological bases. We also investigate the particular properties of the topological basis states in these systems. It is found that the ground state is closely related to parameter q which is the meaningful topological parameter.

What we have been discussing in this paper is still an open problem which will require a deal of further investigations. When the number of particle spreads to $2N$-qubit ($N=2,\ldots$) for closed Heisenberg spin chain, we will need to construct more topological bases. This is a work in progress.

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