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# Quantum teleportation and dense coding via topological basis

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**Abstract** By means of Temperley–Lieb Algebra and topological basis, we make a new realization of topological basis, and get sixteen complete orthonormal topological basis states which are all maximally entangled for four quasi-particles. Then we present an explicit protocol for teleporting an arbitrary two-qubit state via a topological basis entanglement channel. We also show that four bits of classical information can be encoded into a topological basis state by two-particle unitary operations.

**Keywords** Quantum teleportation · Dense coding · Topological basis

## 1 Introduction

Quantum teleportation [1], which is one of the most surprising features of quantum mechanics, has been researched by many authors theoretically [2–8] and experimentally [9–11] since Bennett et al. firstly presented the protocol of teleportation. As is known, the standard teleportation protocol of Bennett et al., the state of the qubit $A_1$ to be teleported from Alice to Bob is

$$|\psi\rangle_{A_1} = a|0\rangle_{A_1} + b|1\rangle_{A_1}. \tag{1}$$

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with a, b $\in \mathcal{C}^1$ and $|a|^2 + |b|^2 = 1$, from one place to another, by a sender, Alice, who knows neither the state $|\psi\rangle_{A_1}$ to be teleported nor the location of the intended receiver, Bob. And the entanglement channel between Alice and Bob is one of four maximally entangled Bell states

$$|\Phi_{1,2}\rangle_{A_2B} = \frac{1}{\sqrt{2}}(|00\rangle_{A_2B} \pm |11\rangle_{A_2B})$$
$$|\Phi_{3,4}\rangle_{A_2B} = \frac{1}{\sqrt{2}}(|01\rangle_{A_2B} \pm |10\rangle_{A_2B}) \tag{2}$$

If Alice and Bob use entanglement channel $|\Phi_1\rangle_{A_2B}$, the initial state of the qubits $A_1$, $A_2$ and B can be expressed as

$$|\psi\rangle_{A_1} \otimes |\Phi_1\rangle_{A_2B} = \frac{1}{2}\sum_{i=1}^{4} |\Phi_i\rangle_{A_1A_2} \otimes |\psi_i\rangle_B. \tag{3}$$

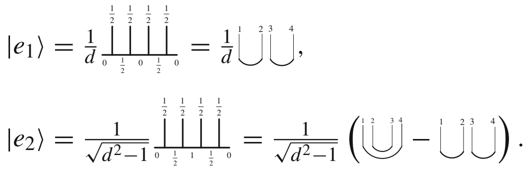
where $|\psi_i\rangle_B = U(i)^{-1}|\psi\rangle_B$, the state $|\psi\rangle_B$ of the qubit B is just the state to be teleported as shown in Eq. (1), and

$$U(1) = I; \quad U(2) = \sigma^z$$
$$U(3) = \sigma^x; \quad U(4) = i\sigma^y \tag{4}$$

Alice and Bob can complete the teleportation through Alice's measurements on the qubits $A_1$ and $A_2$ with four Bell states $|\Phi_i\rangle_{A_1A_2}(i = 1, 2, 3, 4)$ and Bob's unitary rotation $U(i)$ on his qubit B corresponding to the Alice's measurement outcome i, respectively.

Bennett and Wiesner [12] also showed a special feature of Einstein-Podolsky-Rosen (EPR) states (Bell states), i.e. dense coding. It was presented that two bits of classical information can be encoded into an Bell state by one-partical unitary operations. Specifically, if Alice and Bob respectively hold one particle from an Bell state, Alice can send two bits to Bob through performing one of four unitary operations on her particle and transmitting it to Bob. One particle carries two bits of information, which is just the reason why it is called dense coding.

The Topological Quantum Computation (TQC) [13,14] is an intriguing proposal to use the braiding operations of non-Abelian quasiparticles in certain strongly correlated electron systems, also such as the fractional quantum Hall (FQH) liquids and quantum gates. In the topological quantum computation theory, the two-dimensional (2D) braid behavior under the exchange of anyons [15,16] has been investigated based on the $\nu = \frac{5}{2}$ fractional quantum Hall effect (FQHE) [17–21]. The orthonormal topological basis states read [17–20]

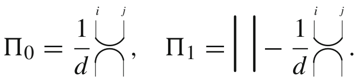


(5)

where the parameter d represents the value of a loop. In the middle fusion chains (called conformal block), at each trivalent vertex the internal edges obey the fusion rules as follows:

$$\frac{1}{2}\times\frac{1}{2}=0+1,\quad \frac{1}{2}\times 1=\frac{1}{2},\quad 1\times 1=0,$$
$$0\times 0=0,\quad 0\times\frac{1}{2}=\frac{1}{2},\quad 0\times 1=1.$$

It is worth noting that there are two different fusion channels for two $\frac{1}{2}$ anyons. From the conformal basis to the Kauffman graph on the right-hand sides, Jones–Wenzl projector operators have been applied, i.e.,



(6)

Recently, it is found that topological basis has some important physical applications in topological quantum computation, quantum entanglement and topological quantum teleportation [22–25]. Based on the topological basis, ref. [22] nested the TLA into the four-dimensional (4D) Yang-Baxter Equation (YBE) and reduced it to the 2D YBE. Then they pointed out that the YBE can be tested in terms of quantum optics. In ref. [24], authors connected the topological basis states with a Heisenberg *XXX* spin chain. On the other hand, an experimental results for a small-scale approximate evaluation of the Jones polynomial by nuclear magnetic resonance (NMR) was presented in ref. [26]. The authors could obtain the value of the Jones polynomial via measuring the nuclear spin state of the molecule.

Our aim in this work is to connect the topological basis states with quantum teleportation and dense coding. In ref. [22], Ge et al. got and used two topological basis states which are not complete for four quasi-particles. In our paper, we will generate sixteen complete orthonormal topological basis states which are all maximally entangled for four quasi-particles. Consequently, we can make use of all of these topological basis states to realize a protocol for teleporting an arbitrary two-qubit state and we show that four bits of classical information can be encoded into a topological basis state by two-particle unitary operation.

This paper is organized as follows: In Sect. 2, we recall the Temperley–Lieb algebra and get part of topological basis states. In Sect. 3, we make a new realization of topological basis, and get sixteen maximally entangled complete orthonormal topological basis states for four quasi-particles. In Sect. 4, we present an explicit protocol for teleporting an arbitrary two-qubit state via a topological basis entanglement channel.

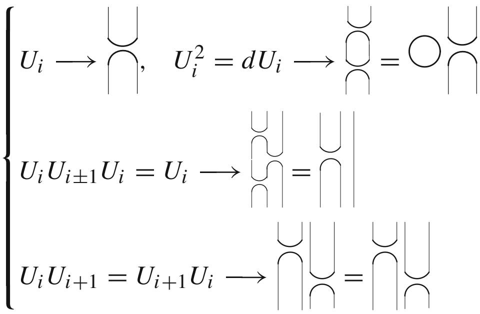
In Sect. 5, we show how the dense coding happens via the topological basis states. We end with a summary.

## 2 Temperley–Lieb algebra and topological basis

We first briefly review the theory of TLA [27]. For each natural number m, the TLA $TL_m(d)$ is generated by $\{I, U_1, U_2 ... U_{m-1}\}$ with the TLA relations:

$$\begin{cases} U_i^2 = dU_i & 1 \leq i \leq m-1 \\ U_i U_{i\pm1} U_i = U_i & 1 \leq i \leq m \\ U_i U_{i+1} = U_{i+1} U_i & \mid i-j \mid \geq 2 \end{cases} \tag{7}$$

where the notation $U_i \equiv U_{i,i+1}$ is used. The $U_i$ represents $1_1 \otimes 1_2 \otimes 1_3 \otimes \cdots \otimes 1_{i-1} \otimes U \otimes 1_{i+2} \cdots 1_m$, and $1_j$ represents the unit matrix in the jth space $V_j$. In addition, the TLA is easily understood in terms of knot diagrams in ref. [26,28,29]. Using Kauffman's graphs, it can be expressed as,



(8)

The $4 \otimes 4$ Temperley–Lieb matrix U with $d = \sqrt{2}$ which satisfies TLA in Eq. (7) has the representation,

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & e^{i\varphi} \\ 0 & 1 & i\varepsilon & 0 \\ 0 & -i\varepsilon & 1 & 0 \\ e^{-i\varphi} & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

where $\varphi$ is real and $\varepsilon = \pm$.

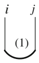
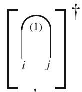
When $d_1 = d_2 = 2$, correspondingly the $4 \otimes 4$ Temperley–Lieb matrices $U^{(1)}$ and $U^{(2)}$ have forms as follows,

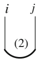
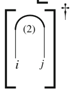
$$U^{(1)} = \begin{pmatrix} 1 & 0 & 0 & e^{i\varphi} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{-i\varphi} & 0 & 0 & 1 \end{pmatrix}, U^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

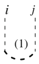
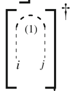
Here the Temperley–Lieb matrices U, $U^{(1)}$ and $U^{(2)}$ all satisfy TLA in Eq. (7). It is easy to see that the Temperley–Lieb matrices U, $U^{(1)}$ and $U^{(2)}$ have the relation as follows:

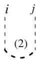
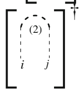
$$U = \frac{1}{\sqrt{2}}\left(U^{(2)} + U^{(1)}\right) \tag{11}$$

For the following convenience, we use the solid lines, dash lines and solid (dash) lines with labels (1), (2) to distinguish different graph states. Then we can introduce a set of $|cup\rangle$ and $\langle cap|$ states and their spin realization as following:

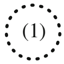
 $= \sqrt{d_1}|\psi_d^{(1)}\rangle_{ij} = \sqrt{\frac{d_1}{2}}\left[|\uparrow\uparrow\rangle_{ij} + e^{-i\varphi}|\downarrow\downarrow\rangle_{ij}\right] = \sqrt{d_1}\left[{}_{ij}\langle\psi_d^{(1)}|\right]^\dagger =$  ,

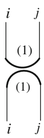
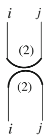
 $= \sqrt{d_2}|\psi_d^{(2)}\rangle_{ij} = \sqrt{\frac{d_2}{2}}\left[|\uparrow\downarrow\rangle_{ij} - i\epsilon|\downarrow\uparrow\rangle_{ij}\right] = \sqrt{d_2}\left[{}_{ij}\langle\psi_d^{(2)}|\right]^\dagger =$  ,

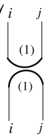
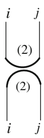
 $= \sqrt{d_1}|\psi_0^{(1)}\rangle_{ij} = \sqrt{\frac{d_1}{2}}\left[|\uparrow\uparrow\rangle_{ij} - e^{-i\varphi}|\downarrow\downarrow\rangle_{ij}\right] = \sqrt{d_1}\left[{}_{ij}\langle\psi_0^{(1)}|\right]^\dagger =$  ,

 $= \sqrt{d_2}|\psi_0^{(2)}\rangle_{ij} = \sqrt{\frac{d_2}{2}}\left[|\uparrow\downarrow\rangle_{ij} + i\epsilon|\downarrow\uparrow\rangle_{ij}\right] = \sqrt{d_2}\left[{}_{ij}\langle\psi_0^{(2)}|\right]^\dagger =$  ,

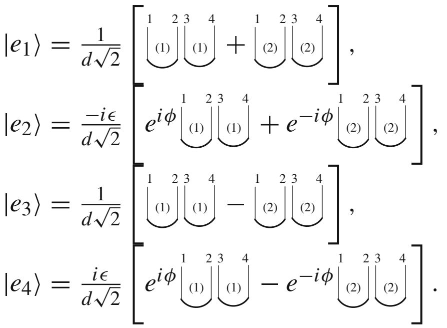
(12)

where the notation $\uparrow$ ($\downarrow$) denotes spin-up(spin-down) corresponding to 0(1) as above, and the notation $|\alpha\beta\rangle_{ij}$ is the abbreviated form of $|\alpha\rangle_i \otimes |\beta\rangle_j (\alpha, \beta = \uparrow, \downarrow)$. The topological parameter(the single loop) $d_1 =$  $=$  $= 2, d_2 =$  $=$  $= 2$. In terms of CAP-CUP language, the T-L matrix in Eqs. (9), (10) can be recast as following,

$U_{ij}^{(1)} =$  , $U_{ij}^{(2)} =$  (13)

$U_{ij} = \frac{1}{\sqrt{2}}\Bigg($  $+$  $\Bigg)$. (14)

Following Ge et al. [22], to reduce the $4 \otimes 4$ Temperley–Lieb matrix, we can introduce a set of topological basis states for four quasi-particles. The topological basis states have the following form,



(15)

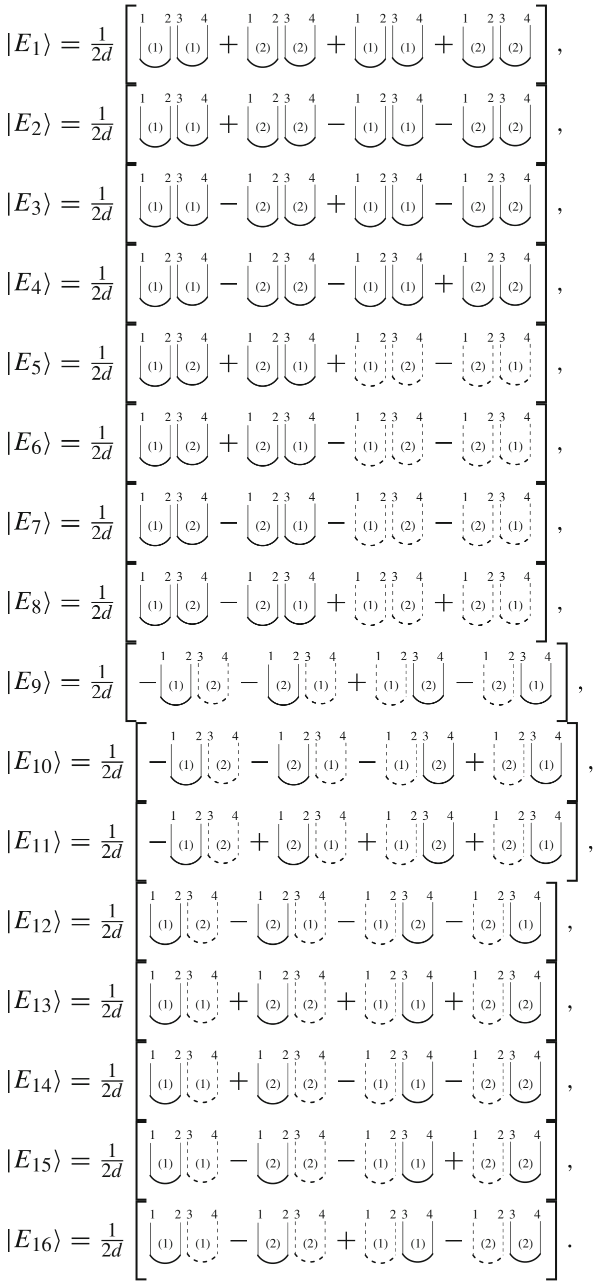
This set of topological basis states are orthonormal basis, (i.e. $\langle e_i|e_j\rangle = \delta_{ij}$). By means of Eqs. (12), (13) and (15), we can verify that the Temperley–Lieb matrix can be reduced to two identical 2-dimensional representations through topological calculation. The basis of subspace are $\{|e_1\rangle, |e_2\rangle\}$ and $\{|e_3\rangle, |e_4\rangle\}$. The 2D representations on the subspace $\{|e_1\rangle, |e_2\rangle\}$ ($\{|e_3\rangle, |e_4\rangle\}$) have the following form:

$$U_A = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}; U_B = \begin{pmatrix} d^{-1} & \sqrt{1-d^{-2}} \\ \sqrt{1-d^{-2}} & d-d^{-1} \end{pmatrix} \tag{16}$$

where $(U_A)_{ij} = \langle e_i|U_{12}|e_j\rangle$ and $(U_B)_{ij} = \langle e_i|U_{23}|e_j\rangle$ (i,j = 1,2). We can verify that $U_A$ and $U_B$ also satisfy the 2D Temperley–Lieb relations, $U_A^2 = dU_A, U_B^2 = dU_B, U_A U_B U_A = U_A$ and $U_B U_A U_B = U_B$.

## 3 A new realization of orthogonal complete topological basis

As Sect. 2, we have got four orthonormal topological basis which are not complete for four quasi-particles. On the subspace $\{|e_1\rangle, |e_2\rangle\}$ and the subspace $\{|e_3\rangle, |e_4\rangle\}$ respectively, the Temperley–Lieb matrix can be reduced to two identical 2-dimensional representations while the subspace representation $U_A$ is diagonal and the $U_B$ is non-diagonal which are same as the usual result in Ge et al. [22]. Based on this, we make a new realization, we let the representation $U_A$ on the subspace is non-diagonal while the $U_B$ is diagonal and let every subspace be minimal, then we generate sixteen complete orthonormal topological basis states through topological calculation:



(17)

where for the following convenience, we have let the parameter $\varphi = 0$. Also by means of Eqs. (12), (13) and (17), we can verify that the Temperley–Lieb matrix can be reduced to eight identical 2-dimensional representations through topological calculation. The basis of subspace are $\{|E_i\rangle, |E_{i+1}\rangle\}$, $(i = 1, 3, 5 \ldots 15)$. The 2D representations on the subspace $\{|E_i\rangle, |E_{i+1}\rangle\}$ have the following form:

$$\mathbb{U}_A = \begin{pmatrix} d^{-1} & \sqrt{1-d^{-2}} \\ \sqrt{1-d^{-2}} & d-d^{-1} \end{pmatrix}; \quad \mathbb{U}_B = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} \tag{18}$$

It is worth noting that we get eight 2-dimensional subspaces, and the corresponding 2D representations on the subspace are all uniform while the 2D Temperley–Lieb matrix representation $\mathbb{U}_A = U_B$ is diagonal and $\mathbb{U}_B = U_A$ is non-diagonal. It is easy to verify that $\mathbb{U}_A$ and $\mathbb{U}_B$ also satisfy the 2D Temperley–Lieb relations, $\mathbb{U}_A^2 = d\mathbb{U}_A$, $\mathbb{U}_B^2 = d\mathbb{U}_B$, $\mathbb{U}_A\mathbb{U}_B\mathbb{U}_A = \mathbb{U}_A$ and $\mathbb{U}_B\mathbb{U}_A\mathbb{U}_B = \mathbb{U}_B$.

According to Eqs. (12) and (17), we can get the spin realization of these topological basis states as follows:

$$\begin{aligned}
|E_1\rangle &= \frac{1}{2}(|\uparrow\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\downarrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle) \\
|E_2\rangle &= \frac{1}{2}(|\uparrow\uparrow\downarrow\downarrow\rangle - i\varepsilon|\uparrow\downarrow\downarrow\uparrow\rangle - i\varepsilon|\downarrow\uparrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle) \\
|E_3\rangle &= \frac{1}{2}(|\uparrow\uparrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle) \\
|E_4\rangle &= \frac{1}{2}(|\uparrow\uparrow\downarrow\downarrow\rangle + i\varepsilon|\uparrow\downarrow\downarrow\uparrow\rangle + i\varepsilon|\downarrow\uparrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle) \\
|E_5\rangle &= \frac{1}{2}(|\uparrow\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\downarrow\rangle - i\varepsilon|\downarrow\uparrow\uparrow\uparrow\rangle - i\varepsilon|\downarrow\downarrow\downarrow\uparrow\rangle) \\
|E_6\rangle &= \frac{1}{2}(-i\varepsilon|\uparrow\uparrow\downarrow\uparrow\rangle + |\uparrow\downarrow\uparrow\uparrow\rangle - i\varepsilon|\downarrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\downarrow\rangle) \\
|E_7\rangle &= \frac{1}{2}(-i\varepsilon|\uparrow\uparrow\downarrow\uparrow\rangle - |\uparrow\downarrow\uparrow\uparrow\rangle + i\varepsilon|\downarrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\downarrow\rangle) \\
|E_8\rangle &= \frac{1}{2}(|\uparrow\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\downarrow\rangle + i\varepsilon|\downarrow\uparrow\uparrow\uparrow\rangle - i\varepsilon|\downarrow\downarrow\downarrow\uparrow\rangle) \\
|E_9\rangle &= \frac{1}{2}(-i\varepsilon|\uparrow\uparrow\downarrow\uparrow\rangle - |\uparrow\downarrow\uparrow\uparrow\rangle - i\varepsilon|\downarrow\uparrow\downarrow\downarrow\rangle - |\downarrow\downarrow\uparrow\downarrow\rangle) \\
|E_{10}\rangle &= \frac{1}{2}(-|\uparrow\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\downarrow\rangle + i\varepsilon|\downarrow\uparrow\uparrow\uparrow\rangle - i\varepsilon|\downarrow\downarrow\downarrow\uparrow\rangle) \\
|E_{11}\rangle &= \frac{1}{2}(|\uparrow\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\downarrow\rangle + i\varepsilon|\downarrow\uparrow\uparrow\uparrow\rangle + i\varepsilon|\downarrow\downarrow\downarrow\uparrow\rangle) \\
|E_{12}\rangle &= \frac{1}{2}(i\varepsilon|\uparrow\uparrow\downarrow\uparrow\rangle - |\uparrow\downarrow\uparrow\uparrow\rangle - i\varepsilon|\downarrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\downarrow\rangle) \\
|E_{13}\rangle &= \frac{1}{2}(|\uparrow\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle - |\downarrow\downarrow\downarrow\downarrow\rangle) \\
|E_{14}\rangle &= \frac{1}{2}(-|\uparrow\uparrow\downarrow\downarrow\rangle + i\varepsilon|\uparrow\downarrow\downarrow\uparrow\rangle - i\varepsilon|\downarrow\uparrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle) \\
|E_{15}\rangle &= \frac{1}{2}(|-\uparrow\uparrow\downarrow\downarrow\rangle - i\varepsilon|\uparrow\downarrow\downarrow\uparrow\rangle + i\varepsilon|\downarrow\uparrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle) \\
|E_{16}\rangle &= \frac{1}{2}(|\uparrow\uparrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\downarrow\uparrow\rangle - |\downarrow\downarrow\downarrow\downarrow\rangle)
\end{aligned} \tag{19}$$

we can see that these topological basis states are not reducible to a pair of Bell states, it means that these topological basis states are genuine four-qubit entangled states. In ref. [30,31], the collective measures of multipartite entanglement are given as,

$$Q = 2 - \frac{2}{n_q} \sum_{i=1}^{n_q} \mathrm{Tr}[\rho_i^2]. \tag{20}$$

According to Eqs. (19) and (20), we can get $Q_i = 1$, $(i = 1, 2, 3 \ldots 16)$, so we get sixteen maximal entangled complete orthonormal topological basis states.

## 4 Teleporting an arbitrary two-qubit state via a topological basis entanglement channel

In this section, we will present an explicit protocol for teleporting an arbitrary two-qubit state $|\psi\rangle_{A_1A_2} = a|\uparrow\uparrow\rangle + b|\uparrow\downarrow\rangle + c|\downarrow\uparrow\rangle + d|\downarrow\downarrow\rangle$ from Alice to Bob via one of sixteen maximally entangled topological basis states. We choose the topological basis state $|E_1\rangle$ in Eq. (19) as the entanglement channel $|\Phi\rangle_{A_3A_4B_1B_2}$ between Alice and Bob. Alice has original particles $A_1A_2$ whose unknown state $|\psi\rangle_{A_1A_2}$ she seeks to teleport to Bob while Alice and Bob share a priori two pairs of particles, $A_3A_4$ and $B_1B_2$, the entanglement channel is the state,

$$\begin{aligned} |\Phi\rangle_{A_3A_4B_1B_2} &= |E_1\rangle_{A_3A_4B_1B_2} \\ &= \frac{1}{2}(|\uparrow\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\downarrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle) \end{aligned} \tag{21}$$

thus the initial complete state of the six particles, $A_1$, $A_2$, $A_3$, $A_4$,$B_1$ and $B_2$ is,

$$|\psi\rangle_{A_1A_2} \otimes |\Phi\rangle_{A_3A_4B_1B_2} = |\psi\rangle_{A_1A_2} \otimes |E_1\rangle_{A_3A_4B_1B_2} \tag{22}$$

It is a pure product state. If Alice performs a complete projective measurement jointly on $A_1A_2A_3A_4$ in the above maximally entangled complete orthonormal topological basis in Eq. (19), then we obtain

$$|\psi\rangle_{A_1A_2} \otimes |E_1\rangle_{A_3A_4B_1B_2} = \frac{1}{4} \sum_{i=1}^{16} |E_i\rangle_{A_1A_2A_3A_4} \otimes |\psi_i\rangle_{B_1B_2} \tag{23}$$

where $|\psi_i\rangle_{B_1B_2} = U(i)^{-1}|\psi\rangle_{B_1B_2}$, the state $|\psi\rangle_{B_1B_2}$ of particles $B_1$, $B_2$ is just the state to be teleported as $|\psi\rangle_{A_1A_2}$, and the unitary rotation operations,

$$\begin{aligned} &U(1) = I_1 \otimes I_2; \quad U(2) = \sigma_1^z \otimes \sigma_2^z \\ &U(3) = \sigma_1^z \otimes I_2; \quad U(4) = I_1 \otimes \sigma_2^z \\ &U(5) = \frac{1}{2} N_1 \otimes M_2^+; \quad U(6) = \frac{1}{2} M_1 \otimes N_2^+ \end{aligned}$$

$$
\begin{aligned}
U(7) &= \frac{i}{2} M_1 \otimes M_2^+; \quad U(8) = -\frac{i}{2} N_1 \otimes N_2^+ \\
U(9) &= \frac{1}{2}\left(N_1^+ \otimes I_2 + I_1 \otimes M_2 - M_1 \otimes \sigma_2^z + \sigma_1^z \otimes N_2^+\right) \\
U(10) &= \frac{1}{2}\left(-N_1^+ \otimes I_2 + I_1 \otimes M_2 + M_1 \otimes \sigma_2^z + \sigma_1^z \otimes N_2^+\right) \\
U(11) &= \frac{1}{2}\left(M_1 \otimes I_2 - I_1 \otimes N_2^+ - N_1^+ \otimes \sigma_2^z - \sigma_1^z \otimes M_2\right) \\
U(12) &= \frac{1}{2}\left(M_1 \otimes I_2 + I_1 \otimes N_2^+ - N_1^+ \otimes \sigma_2^z + \sigma_1^z \otimes M_2\right) \\
U(13) &= \frac{1}{2}\left(N_1 \otimes I_2 + I_1 \otimes M_2^+ - M_1^+ \otimes \sigma_2^z + \sigma_1^z \otimes N_2\right) \\
U(14) &= \frac{1}{2}\left(N_1 \otimes I_2 - I_1 \otimes M_2^+ - M_1^+ \otimes \sigma_2^z - \sigma_1^z \otimes N_2\right) \\
U(15) &= \frac{1}{2}\left(-M_1^+ \otimes I_2 - I_1 \otimes N_2 + N_1 \otimes \sigma_2^z - \sigma_1^z \otimes M_2^+\right) \\
U(16) &= \frac{1}{2}\left(M_1^+ \otimes I_2 - I_1 \otimes N_2 - N_1 \otimes \sigma_2^z - \sigma_1^z \otimes M_2^+\right)
\end{aligned} \tag{24}
$$

where $M_1 = e^{i\frac{\pi}{4}}(\sigma_1^x + \sigma_1^y)$, $N_1 = e^{i\frac{\pi}{4}}(\sigma_1^x - \sigma_1^y)$, $M_2 = e^{i\frac{\pi}{4}}(\sigma_2^x + \sigma_2^y)$, $N_2 = e^{i\frac{\pi}{4}}(\sigma_2^x - \sigma_2^y)$, we have let $\varepsilon = 1$ for convenience. It follows that, regardless of the unknown state $|\psi\rangle_{A_1A_2}$, the sixteen measurement outcomes are equally likely, each occurring with the probability $1/16$. Alice gains no information about the state $|\psi\rangle_{A_1A_2}$ from her measurement. She is left with particles $A_1, A_2, A_3, A_4$ in some maximally entangled topological basis states, without any trace of the original $|\psi\rangle_{A_1A_2}$. The outcome of Alice's measurement constitutes the second purely classical part of the full information encoded in $|\psi\rangle_{A_1A_2}$. Thus an accurate teleportation can be achieved in all cases by having Alice tell Bob the classical outcome of her measurement on the qubits $A_1, A_2, A_3, A_4$ with sixteen topological basis states $|E_i\rangle_{A_1A_2A_3A_4}(i = 1, 2, 3, 4 \ldots 16)$, after which Bob applies the corresponding unitary rotation operation to transform the state $|\psi_i\rangle_{B_1B_2}$ of his particles $B_1, B_2$ into an accurate replica of the original state $|\psi\rangle_{A_1A_2}$ of Alice's particles $A_1A_2$.

## 5 Dense coding via topological basis

As Eqs. (17) and (19) in Sect. 3, we have got sixteen maximal entangled complete orthonormal topological basis states. We will show how the dense coding happens via the topological basis states in this section.

Here we let $|E_i\rangle = |E_i\rangle_{1234}$, $(i = 1, 2, 3 \ldots 16)$, where the subscripts 1,2,3,4 denote different particles. These states are orthonormal with each other and constitute a complete basis, i.e topological basis. We get sixteen two-particle unitary operations (for particles 1 and 2): $I_1I_2$, $I_1\sigma_2^x$, $iI_1\sigma_2^y$, $I_1\sigma_2^z$, $-\sigma_1^x\sigma_2^x$, $i\sigma_1^x\sigma_2^y$, $-\sigma_1^x\sigma_2^z$, $-\sigma_1^x I_2$, $-i\sigma_1^y\sigma_2^x$, $-\sigma_1^y\sigma_2^y$, $-i\sigma_1^y\sigma_2^z$, $-i\sigma_1^y I_2$, $\sigma_1^z\sigma_2^x$, $i\sigma_1^z\sigma_2^y$, $\sigma_1^z\sigma_2^z$, $\sigma_1^z I_2$, where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{25}$$

Without loss of generality, suppose Alice and Bob share an topological basis state $|E_5\rangle$, that is, Alice has particles 1,2 and Bob holds 3,4. Alice can encode four bits of information into the state by performing one of the above sixteen operations on particales1,2, under which this state changes as,

$$\begin{aligned}
&I_1 I_2 |E_5\rangle \to |E_5\rangle; \quad I_1 \sigma_2^x |E_5\rangle \to |E_3\rangle; \quad i I_1 \sigma_2^y |E_5\rangle \to |E_2\rangle; \quad I_1 \sigma_2^z |E_5\rangle \to |E_7\rangle \\
&\quad -\sigma_1^x \sigma_2^x |E_5\rangle \to |E_{10}\rangle; \quad i\sigma_1^x \sigma_2^y |E_5\rangle \to |E_{12}\rangle; \quad -\sigma_1^x \sigma_2^z |E_5\rangle \to |E_{13}\rangle; \\
&\quad -\sigma_1^x I_2 |E_5\rangle \to |E_{16}\rangle \quad -i\sigma_1^y \sigma_2^x |E_5\rangle \to |E_{11}\rangle; \\
&\quad -\sigma_1^y \sigma_2^y |E_5\rangle \to |E_9\rangle; \quad -i\sigma_1^y \sigma_2^z |E_5\rangle \to |E_{15}\rangle; \\
&\quad -i\sigma_1^y I_2 |E_5\rangle \to |E_{14}\rangle \quad \sigma_1^z \sigma_2^x |E_5\rangle \to |E_1\rangle; \\
&\; i\sigma_1^z \sigma_2^y |E_5\rangle \to |E_4\rangle; \quad \sigma_1^z \sigma_2^z |E_5\rangle \to |E_6\rangle; \quad \sigma_1^z I_2 |E_5\rangle \to |E_8\rangle
\end{aligned} \tag{26}$$

where the superscript of these operations represents the qubit on which the operations are performed. Afterwards Alice sends particles 1,2 to Bob. Bob can distinguish which operation is chosen by Alice via a topological basis measurement on particles 1,2 and 3,4. If $I_1 I_2, I_1\sigma_2^x, iI_1\sigma_2^y, I_1\sigma_2^z, -\sigma_1^x\sigma_2^x, i\sigma_1^x\sigma_2^y, -\sigma_1^x\sigma_2^z, -\sigma_1^x I_2, -i\sigma_1^y\sigma_2^x, -\sigma_1^y\sigma_2^y,$ $-i\sigma_1^y\sigma_2^z, -i\sigma_1^y I_2, \sigma_1^z\sigma_2^x, i\sigma_1^z\sigma_2^y, \sigma_1^z\sigma_2^z, \sigma_1^z I_2$ represent 0000, 0001, 0010, 0011, 0100, 0101, 0110, 01111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111 respectively, Bob can obtain four bits from Alice. For example, Bob knows Alice's message is 1000 if his measurement result is $|E_{11}\rangle$. Similarly, any one of the sixteen topological basis states in Eqs. (17) and (19) can be used as the original state in this communication. It is worth noting that we can also get operations for any two particles in particles 1,2,3,4, to realize the dense coding via the topological basis states. It means that Alice can encode much more classical information into a topological basis state by choosing unitary operation for different two particles.

## 6 Summary

In summary, by means of Temperley–Lieb Algebra(TLA) and topological basis, we have made a new realization of topological basis, and have got sixteen complete orthonormal topological basis which are genuine four-partite entangled states and are all maximally entangled for four quasi-particles. Based on this, faithful teleportation of an arbitrary two-qubit state via a topological basis entanglement channel has been studied in detail. We also show that four bits of classical information can be encoded into a topological basis state by two-particle unitary operation.

Eventually, people have currently found that topological basis has some important physical applications in topological quantum computation and quantum entanglement, how to reveal the role of the topological parameter d is also an interesting and significant topic. We shall investigate this subject subsequently.

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