Propagation equation of Hermite–Gauss beams through a complex optical system with apertures and its application to focal shift

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Based on the generalized Huygens–Fresnel diffraction integral (Collins' formula), the propagation equation of Hermite–Gauss beams through a complex optical system with a limiting aperture is derived. The elements of the optical system may be all those characterized by an *ABCD* ray-transfer matrix, as well as any kind of apertures represented by complex transmittance functions. To obtain the analytical expression, we expand the aperture transmittance function into a finite sum of complex Gaussian functions. Thus the limiting aperture is expressed as a superposition of a series of Gaussian-shaped limiting apertures. The advantage of this treatment is that we can treat almost all kinds of apertures in theory. As application, we define the width of the beam and the focal plane using an encircled-energy criterion and calculate the intensity distribution of Hermite–Gauss beams at the actual focus of an aperture lens.

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1. INTRODUCTION

In the generation and propagation of a laser there exist more or less limiting apertures. Therefore, the beam propagation through the optical system with a limiting aperture is of practical importance. In general, the direct numerical integration of Collins' formula is necessary for studying the propagation and transformation of laser beams through the optical system with limiting apertures, but it is often cumbersome and time consuming. Thus a large effort has been devoted to developing all kinds of methods to simplify the calculation process. The matrix representation [1] and recurrence algorithm [2] have been derived to reach this goal. The matrix representation in [1] holds only for circularly symmetric optical systems and laser beams. The recurrence algorithm in [2] is required to repeat the recursive procedures to derive the propagation expression for the beams of high order. In this paper, a propagation equation of Hermite–Gauss beams through the complex optical system with limiting apertures is derived for the purpose of fast computation, and the intensity distribution of Hermite–Gauss beams at the actual focus of an aperture lens is calculated by means of the equation. This method can hold for all kinds of systems characterized by *ABCD* ray-transfer matrices and obtain the propagation expression directly for the given beams without recursion.

The outline of this paper is as follows: First, we derive the analytical equation for the beam propagation. Then, the propagation equation is applied to an aperture lens. Finally, we discuss the advantages of this method and give the suggestions for achieving sufficient accuracy.

2. BEAM PROPAGATION THROUGH OPTICAL SYSTEMS WITHOUT APERTURES

The field distribution of laser beams, which is transverse to the direction of propagation, may be described by the Hermite–Gauss function and is of the form [3] as follows:

$$U_i(x_i, y_i) = K_i \exp[-j(\pi/\lambda q_{ix})x_i^2] H_m\left(\sqrt{2}\frac{x_i}{\omega_{ix}}\right) \times \exp[-j(\pi/\lambda q_{iy})y_i^2] H_n\left(\sqrt{2}\frac{y_i}{\omega_{iy}}\right), \quad (1)$$

where

$$\frac{1}{q_{ix,y}} = \frac{1}{R_{ix,y}} - j\frac{\lambda}{\pi\omega_{ix,y}^2}, \quad (2)$$

ω_{ix} and ω_{iy} are the $1/e$ field spot radii along the x and y axes, respectively, $R_{ix,y}$ is the wavefront radius of curvature, $q_{ix,y}$ is the beam parameter, H_m is the Hermite polynomials of order m, K_i is a complex constant, and λ is the wavelength.

Let us consider the optical system without apertures shown in Fig. 1. F_I and F_o are the input and output planes, $E_1, E_2, \ldots, E_N$ are optical elements that can be represented by *ABCD* ray-transfer matrices, and $Z_1, Z_2, \ldots, Z_{N+1}$ are the optical distances between the optical elements. A Gaussian beam may be injected into the system through the input plane F_I. U_i and U are the field distribution at the input and output planes, respectively. U_i is given by Eq. (1). The field distribution U can be obtained by substituting Eq. (1) into Collins' formula [4] as follows:

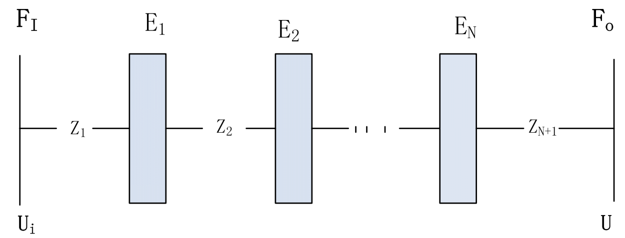

Fig. 1. Schematic representation of an optical system without apertures.

$$\begin{aligned} U(x,y) = {} & K_i \frac{jk}{2\pi\sqrt{B_x B_y}} \exp(-jkL) \int_{-\infty}^{\infty} \exp[-j(\pi/\lambda q_{ix})x_i^2] \\ & \times H_m\left(\sqrt{2}\frac{x_i}{\omega_{ix}}\right) \exp[-j(\pi/\lambda B_x)] \\ & \times [D_x x^2 - 2xx_i + A_x x_i^2]\mathrm{d}x_i \\ & \times \int_{-\infty}^{\infty} \exp[-j(\pi/\lambda q_{iy})y_i^2] H_n\left(\sqrt{2}\frac{y_i}{\omega_{iy}}\right) \exp[-j(\pi/\lambda B_y)] \\ & \times [D_y y^2 - 2yy_i + A_y y_i^2]\mathrm{d}y_i, \end{aligned} \tag{3}$$

where k is the wavenumber, L is the optical distance along the z axis, and $A_{x,y}$, $B_{x,y}$, and $D_{x,y}$ are the elements of the x- and y-axis ray-transfer matrix, respectively.

$U(x,y)$ consists of a product of two functions f and g. f depends only upon x and g only upon y. Thus we can write Eq. (3) in the following way:

$$U(x,y) = K_i f(x) g(y), \tag{4}$$

$$\begin{aligned} f(x) = {} & \left[\frac{jk}{2\pi B_x} \exp(-jkL)\right]^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp[-j(\pi/\lambda q_{ix})x_i^2] H_m\left(\sqrt{2}\frac{x_i}{\omega_{ix}}\right) \\ & \times \exp[-j(\pi/\lambda B_x)][D_x x^2 - 2xx_i + A_x x_i^2]\mathrm{d}x_i, \end{aligned} \tag{5}$$

$$\begin{aligned} g(y) = {} & \left[\frac{jk}{2\pi B_y} \exp(-jkL)\right]^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp[-j(\pi/\lambda q_{iy})y_i^2] H_n\left(\sqrt{2}\frac{y_i}{\omega_{iy}}\right) \\ & \times \exp[-j(\pi/\lambda B_y)][D_y y^2 - 2yy_i + A_y y_i^2]\mathrm{d}y_i. \end{aligned} \tag{6}$$

Because both functions $f(x)$ and $g(y)$ contain integrals of the same kind, we can solve the integral only in function $f(x)$, and then function $g(y)$ is obviously known.

With the aid of integral formula [5]

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp[-(y-t)^2] H_m\left(\frac{t}{\lambda}\right)\mathrm{d}t = \left(1 - \frac{1}{\lambda^2}\right)^{m/2} H_m\left(\frac{y}{\sqrt{\lambda^2 - 1}}\right), \tag{7}$$

we obtain the solution as follows:

$$\begin{aligned} f(x) = {} & \left[\frac{\omega_{ix}}{\omega_x} \exp(-jkL)\right]^{1/2} \exp\left[-j\left(\frac{1}{2} + m\right)\delta_x\right] \\ & \times \exp[-j(\pi/\lambda q_x)x^2] H_m\left(\sqrt{2}\frac{x}{\omega_x}\right). \end{aligned} \tag{8}$$

Owing to the similarity of integrals for both functions $f(x)$ and $g(y)$, $g(y)$ is given by

$$\begin{aligned} g(y) = {} & \left[\frac{\omega_{iy}}{\omega_y} \exp(-jkL)\right]^{1/2} \exp\left[-j\left(\frac{1}{2} + n\right)\delta_y\right] \\ & \times \exp[-j(\pi/\lambda q_y)y^2] H_n\left(\sqrt{2}\frac{y}{\omega_y}\right), \end{aligned} \tag{9}$$

where

$$q_{x,y} = \frac{A_{x,y} q_{ix,y} + B_{x,y}}{C_{x,y} q_{ix,y} + D_{x,y}}, \qquad \frac{1}{q_{ix,y}} = \frac{1}{R_{ix,y}} - j\frac{\lambda}{\pi\omega_{ix,y}^2}, \tag{10}$$

$$\frac{1}{q_{x,y}} = \frac{1}{R_{x,y}} - j\frac{\lambda}{\pi\omega_{x,y}^2}, \tag{11}$$

$$\frac{\omega_{x,y}}{\omega_{ix,y}} \exp(j\delta_{x,y}) = A_{x,y} + \frac{B_{x,y}}{q_{ix,y}}, \tag{12}$$

$$\delta_{x,y} = \arctan\left[-\frac{\lambda B_{x,y}}{\pi\omega_{ix,y}^2 (A_{x,y} + B_{x,y}/R_{ix,y})}\right], \tag{13}$$

$$\begin{aligned} \frac{1}{R_{x,y}} = {} & \left(\frac{\omega_{ix,y}}{\omega_{x,y}}\right)^2 \left[A_{x,y}C_{x,y} + \frac{1}{R_{ix,y}}(A_{x,y}D_{x,y} + B_{x,y}C_{x,y}) \right. \\ & \left. + \frac{B_{x,y}D_{x,y}}{q_{ix,y}q_{ix,y}^*}\right]. \end{aligned} \tag{14}$$

Equations (4), (8), and (9) yield the field distribution on the output plane F_o,

$$\begin{aligned} U(x,y) = {} & K_i \left(\frac{\omega_{ix}\omega_{iy}}{\omega_x \omega_y}\right)^{1/2} \\ & \times \exp\left\{-j\left[kL + \left(\frac{1}{2} + m\right)\delta_x + \left(\frac{1}{2} + n\right)\delta_y\right]\right\} \\ & \times \exp[-j(\pi/\lambda q_x)x^2] H_m\left(\sqrt{2}\frac{x}{\omega_x}\right) \\ & \times \exp[-j(\pi/\lambda q_y)y^2] H_n\left(\sqrt{2}\frac{y}{\omega_y}\right). \end{aligned} \tag{15}$$

3. LIMITING APERTURES

Suppose that at some location along the optical axis of an optical system there exists an aperture depicted as in Fig. 2. A is the limiting aperture represented by the complex transmittance function. We denote by A_I, B_I, C_I, and D_I the four elements of the M_I matrix describing the Ith set of optical elements separated by aperture A. In such a case, we should find the field at the plane of aperture, multiply it by the corresponding aperture complex transmittance function, and apply Collins' formula a second time to describe propagation through the remainder of the system. The procedure is repeated until the field is computed at the output plane.

For this special case, we obtain the analytical expression by expanding the aperture transmittance function into a finite sum of complex Gaussian functions [6]. The expansion is given by

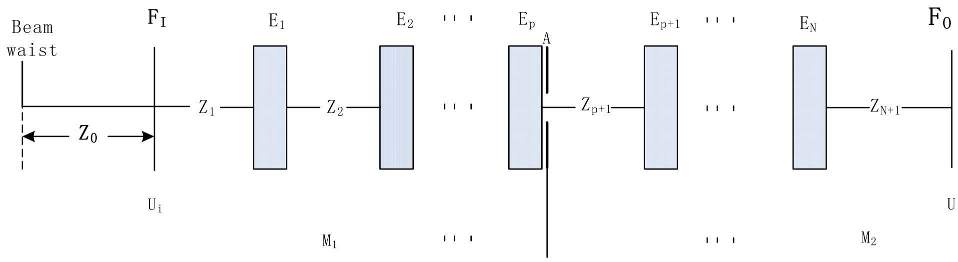

Fig. 2. Schematic representation of an optical system with limiting apertures.

$$T = T(x)T(y) = \sum_{k=1}^{M} u_k \exp(-\varepsilon_k x^2) \sum_{l=1}^{N} v_l \exp(-\varphi_l y^2), \quad (16)$$

where T is the aperture transmittance function, u_k, v_l, ε_k, and φ_l are complex constants, and M and N are the number of Gaussian functions needed. Thus the limiting apertures can be expressed as a superposition of a series of Gaussian-shaped limiting apertures.

To solve the field U at the output plane, we should find the field at the plane of aperture A at first. With the aid of Eq. (15), the field distribution at aperture plane A is

$$U_A(x,y) = K_A \exp[-j(\pi/\lambda q_{Ax})x^2] H_m\left(\sqrt{2}\frac{x}{\omega_{Ax}}\right) \times \exp[-j(\pi/\lambda q_{Ay})y^2] H_n\left(\sqrt{2}\frac{y}{\omega_{Ay}}\right), \quad (17)$$

where

$$K_A = K_i\left(\frac{\omega_{ix}\omega_{iy}}{\omega_{Ax}\omega_{Ay}}\right)^{1/2} \times \exp\left\{-j\left[kL_1 + \left(\frac{1}{2}+m\right)\delta_{Ax} + \left(\frac{1}{2}+n\right)\delta_{Ay}\right]\right\}, \quad (18)$$

$$q_{Ax,y} = \frac{A_{1x,y}q_{ix,y} + B_{1x,y}}{C_{1x,y}q_{ix,y} + D_{1x,y}}, \qquad \frac{1}{q_{ix,y}} = \frac{1}{R_{ix,y}} - j\frac{\lambda}{\pi\omega_{ix,y}^2}, \quad (19)$$

$$\frac{1}{q_{Ax,y}} = \frac{1}{R_{Ax,y}} - j\frac{\lambda}{\pi\omega_{Ax,y}^2}, \quad (20)$$

$$\frac{\omega_{Ax,y}}{\omega_{ix,y}} \exp(j\delta_{Ax,y}) = A_{1x,y} + \frac{B_{1x,y}}{q_{ix,y}}, \quad (21)$$

$$\delta_{Ax,y} = \arctan\left[-\frac{\lambda B_{1x,y}}{\pi\omega_{ix,y}^2(A_{1x,y} + B_{1x,y}/R_{ix,y})}\right], \quad (22)$$

$$\frac{1}{R_{Ax,y}} = \left(\frac{\omega_{ix,y}}{\omega_{Ax,y}}\right)^2\left[A_{1x,y}C_{1x,y} + \frac{1}{R_{ix,y}}(A_{1x,y}D_{1x,y} + B_{1x,y}C_{1x,y}) + \frac{B_{1x,y}D_{1x,y}}{q_{ix,y}q_{ix,y}^*}\right], \quad (23)$$

$$M_{1x,y} = \begin{bmatrix} A_{1x,y} & B_{1x,y} \\ C_{1x,y} & D_{1x,y} \end{bmatrix}, \quad (24)$$

and L_1 is the optical distance between the input plane and aperture A.

Then we multiply U_A by the corresponding aperture complex transmittance function, and apply Collins' formula a second time to obtain the field distribution U at the output plane F_o,

$$U(x,y) = K_A \exp(-jkL_2) \sum_{k=1}^{M} u_k\left(\frac{jk}{2\pi B_{2x}}\right)^{1/2} \int_{-\infty}^{\infty} \exp[-(\varepsilon_k + j\pi/\lambda q_{Ax})x_A^2] H_m\left(\sqrt{2}\frac{x_A}{\omega_{Ax}}\right) \times \exp[-j(\pi/\lambda B_{2x})][D_{2x}x^2 - 2xx_A + A_{2x}x_A^2]\mathrm{d}x_A \times \sum_{l=1}^{N} v_l\left(\frac{jk}{2\pi B_{2y}}\right)^{1/2} \int_{-\infty}^{\infty} \exp[-(\varphi_l + j\pi/\lambda q_{Ax})y_A^2] H_n\left(\sqrt{2}\frac{y_A}{\omega_{Ay}}\right) \times \exp[-j(\pi/\lambda B_{2y})][D_{2y}y^2 - 2yy_A + A_{2y}y_A^2]\mathrm{d}y_A, \quad (25)$$

where L_2 is the optical distance between aperture A and the output plane.

Owing to the similarity of integrals for both Eqs. (3) and (25), we can obtain the solution easily by analogy. Performing the integration yields

$$U(x,y) = K \exp\left(-j\frac{\pi D_{2x}}{\lambda B_{2x}}x^2\right) \exp\left(-j\frac{\pi D_{2y}}{\lambda B_{2y}}y^2\right) \times \sum_{k=1}^{M} \frac{u_k}{Q_x}\left(1 - \frac{2}{\omega_{Ax}^2 Q_x^2}\right)^{m/2} \times \exp[-(\pi/\lambda B_{2x}Q_x)^2 x^2] H_m(P_x x) \times \sum_{l=1}^{N} \frac{v_l}{Q_y}\left(1 - \frac{2}{\omega_{Ay}^2 Q_y^2}\right)^{n/2} \times \exp[-(\pi/\lambda B_{2y}Q_y)^2 y^2] H_n(P_y y), \quad (26)$$

where

$$Q_x^2 = j\frac{\pi}{\lambda q_{Ax}} + j\frac{\pi A_{2x}}{\lambda B_{2x}} + \varepsilon_k Q_y^2 = j\frac{\pi}{\lambda q_{Ay}} + j\frac{\pi A_{2y}}{\lambda B_{2y}} + \varphi_l, \quad (27)$$

$$P_x = j\frac{\pi}{\lambda B_{2x}Q_x(\omega_{Ax}^2 Q_x^2/2 - 1)^{1/2}}, \quad (28)$$

$$P_y = j\frac{\pi}{\lambda B_{2y} Q_y (\omega_{Ay}^2 Q_y^2/2 - 1)^{1/2}}, \tag{29}$$

$$K = K_A\left(j\frac{k}{2\sqrt{B_{2x}B_{2y}}}\right)\exp(-jkL_2), \tag{30}$$

$$M_{2x,y} = \begin{bmatrix} A_{2x,y} & B_{2x,y} \\ C_{2x,y} & D_{2x,y} \end{bmatrix}. \tag{31}$$

In fact, we can generalize the result to the optical system with multiple apertures by repeating the calculation between aperture planes with the aid of the Eqs. (17), (25), and (26).

4. APPLICATION

In this section, we apply the propagation equation to an aperture lens of focal length f, as shown in Fig. 3. The lens fills an aperture of half-width a. Aperture center O is assumed to be on the axis. The distance from the back focus F of the lens to point P is s. The input plane F_I is located at the aperture, and the output plane F_o is located at point P. The ray-transfer matrix of the optical system is M,

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} -s/f & f+s \\ -1/f & 1 \end{bmatrix}. \tag{32}$$

In this case, the aperture transmittance function is given by $T(x,y)$. The aperture transmittance function $T(x,y)$ can be expanded into a finite sum of complex Gaussian functions. The expression is given by

$$T(x,y) = \sum_{k=1}^{M} u_k \exp\left(-\frac{\varepsilon_k x^2}{a^2}\right)\sum_{l=1}^{N} u_l \exp\left(-\frac{\varepsilon_l y^2}{a^2}\right). \tag{33}$$

The complex constants u_k, u_l, ε_k, and ε_l can be obtained by using a computer program based on optimization theory. The numbers M and N are set to be 10. The set of coefficients is given by Table 1 [6].

A Gaussian beam may be injected into the system through the input plane F_I. E_i is the field distribution at the input plane F_I and is given by Eq. (1). The field distribution E on the output plane F_o can be obtained by substituting Eq. (1) into the integral (25). To simplify the calculation, we suppose that the beam waist is located at the input plane F_I. By setting the parameter,

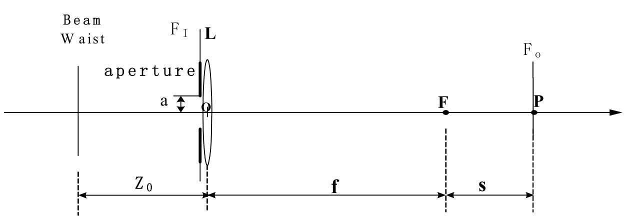

Fig. 3. Schematic representation of the aperture lens.

Table 1. Set of Coefficients u, ε Used in Representing the Aperture Transmittance

N	u	ε
1	$11.428 + 0.95175i$	$4.0697 + 0.22726i$
2	$0.06002 - 0.08013i$	$1.1531 - 20.933i$
3	$-4.2743 - 8.5562i$	$4.4608 + 5.1268i$
4	$1.6576 + 23.7015i$	$4.3521 + 14.997$
5	$-5.0418 + 3.2488i$	$4.5443 + 10.003i$
6	$1.1227 - 0.68854i$	$3.8478 + 20.078i$
7	$-1.0106 - 0.26955i$	$2.5280 - 10.310i$
8	$-2.5974 + 3.2202i$	$3.3197 - 4.8008i$
9	$-0.14840 - 0.31193i$	$1.9002 - 15.820i$
10	$-0.20850 - 0.23851i$	$2.6340 + 25.009i$

$$q_{ix,y} = ik\omega_0^2/2. \tag{34}$$

The field distribution at the input plane F_I is of the form

$$E_i(x_i, y_i) = K_i \exp[-(x_i^2 + y_i^2)/\omega_0^2] H_m\left(\sqrt{2}\frac{x_i}{\omega_0}\right) H_n\left(\sqrt{2}\frac{y_i}{\omega_0}\right). \tag{35}$$

Substituting Eq. (35) into the integral (25) yields, at the output plane,

$$\begin{aligned} E(x,y,z) = {} & K_i \exp(-jkz)\left(j\frac{\pi}{\lambda B}\right)\exp\left(-j\frac{\pi D}{\lambda B}x^2\right) \\ & \times \exp\left(-j\frac{\pi D}{\lambda B}y^2\right)\sum_{k=1}^{M}\frac{u_k}{Q_{x,k}}\left(1-\frac{2}{\omega_0^2 Q_{x,k}^2}\right)^{m/2} \\ & \times \exp\left[-\left(\frac{\pi}{\lambda B Q_{x,k}}\right)^2 x^2\right] H_m(P_{x,k}x) \\ & \times \sum_{l=1}^{N}\frac{u_l}{Q_{y,l}}\left(1-\frac{2}{\omega_0^2 Q_{y,l}^2}\right)^{n/2} \\ & \times \exp\left[-\left(\frac{\pi}{\lambda B Q_{y,l}}\right)^2 y^2\right] H_n(P_{y,l}y), \end{aligned} \tag{36}$$

where

$$Q_{x,k}^2 = \frac{1}{\omega_0^2} + j\frac{\pi A}{\lambda B} + \frac{\varepsilon_k}{a^2} \qquad Q_{y,l}^2 = \frac{1}{\omega_0^2} + j\frac{\pi A}{\lambda B} + \frac{\varepsilon_l}{a^2}, \tag{37}$$

$$P_{x,k} = j\frac{\pi}{\lambda B Q_{x,k}(\omega_0^2 Q_{x,k}^2/2 - 1)^{1/2}}, \tag{38}$$

$$P_{y,l} = j\frac{\pi}{\lambda B Q_{y,l}(\omega_0^2 Q_{y,l}^2/2 - 1)^{1/2}}, \tag{39}$$

and z is the optical distance between the aperture and the output plane.

The intensity distribution $I(x,y,z)$ at the output plane is given by

$$I(x,y,z) = |E(x,y,z)|^2. \tag{40}$$

In usual treatments, the location of the actual focus is defined to be the point at which the beam reaches its maximum on-axis intensity. However, the maximum intensity is not on-axis for all odd mode numbers. So we define the width ω of beam as that radius within which 80% of the beam's energy is enclosed, and the focal plane as the plane at which the beam width is at its smallest value using an encircled-energy criterion [7] as follows:

$$\frac{\int_{-\omega}^{\omega}\int_{-\omega}^{\omega} I(x,y,z)\mathrm{d}x\mathrm{d}y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I(x,y,z)\mathrm{d}x\mathrm{d}y} = 0.8. \tag{41}$$

Because of space limitations, we give only the approximate analytical expression of the beam width of the TEM_{11}-mode Hermite–Gauss beam.

$$S_{11x}S_{11y} = 0.8, \tag{42}$$

where

$$S_{11x,y} = \sum_{k=1}^{10}\sum_{l=1}^{10}\frac{\xi_{x,y}[\sqrt{\pi}erf(\eta_{x,y}\omega) - 2\exp(\eta_{x,y}^2\omega^2)\eta_{x,y}\omega]}{\eta_{x,y}^3} \Big/ \sum_{k=1}^{10}\sum_{l=1}^{10}\frac{\xi_{x,y}\sqrt{\pi}}{\eta_{x,y}^3}, \tag{43}$$

$$\xi_x = \left(\frac{k}{-2B}\right)\frac{u_k u_l^*}{Q_{x,k}Q_{x,l}^*}\left(1-\frac{2}{Q_{x,k}}\right)^{1/2}\left(1-\frac{2}{Q_{x,l}^*}\right)^{1/2} P_{x,k}P_{x,l}^*, \tag{44}$$

$$\xi_y = \left(\frac{k}{-2B}\right)\frac{u_k u_l^*}{Q_{y,k}Q_{y,l}^*}\left(1-\frac{2}{Q_{y,k}}\right)^{1/2}\left(1-\frac{2}{Q_{y,l}^*}\right)^{1/2} P_{y,k}P_{y,l}^*, \tag{45}$$

$$Q_{x,k}^2 = \frac{1}{\omega_0^2} + j\frac{\pi A}{\lambda B} + \frac{\varepsilon_k}{a^2}, \qquad Q_{y,l}^2 = \frac{1}{\omega_0^2} + j\frac{\pi A}{\lambda B} + \frac{\varepsilon_l}{a^2}, \tag{46}$$

$$P_{x,k} = j\frac{\pi}{\lambda B Q_{x,k}(\omega_0^2 Q_{x,k}^2/2 - 1)^{1/2}}, \tag{47}$$

$$P_{y,l} = j\frac{\pi}{\lambda B Q_{y,l}(\omega_0^2 Q_{y,l}^2/2 - 1)^{1/2}}, \tag{48}$$

$$\eta_x^2 = \left(\frac{ikD}{2B} + \frac{k^2}{4B^2Q_{x,k}^2}\right) + \left(\frac{ikD}{2B} + \frac{k^2}{4B^2Q_{x,l}^2}\right)^*, \tag{49}$$

$$\eta_y^2 = \left(\frac{ikD}{2B} + \frac{k^2}{4B^2Q_{y,k}^2}\right) + \left(\frac{ikD}{2B} + \frac{k^2}{4B^2Q_{y,l}^2}\right)^*, \tag{50}$$

$$\mathrm{erf}(y) = \frac{2}{\sqrt{\pi}}\int_0^y \exp(-t^2)\mathrm{d}t. \tag{51}$$

By means of Eqs. (42)–(51), we calculate the actual focus of the aperture lens for the TEM_{11}-mode Hermite–Gauss beam. The calculation parameters are $\lambda = 632.8$ nm, $f = 150$ cm, $\omega_0 = 2$ mm, $a = 1$ mm, and $M = N = 10$. The actual focus is located at 112.338 cm behind the lens and the intensity distribution is shown in Fig. 4.

To demonstrate the time efficiency and precision of our method, we make a comparison to the direct numerical integral in the same environment, including both hardware and software. We calculate the intensity distribution of the TEM_{00}-mode Hermite–Gauss beam through an aperture lens for both methods. The calculation parameters are $\lambda = 632.8$ nm, $f = 150$ cm, $\omega_0 = 2$ mm, $a = 1$ mm, and $M = N = 10$. The intensity distribution located at 112 cm is shown in Fig. 5.

As depicted in Fig. 5, the solid line and asterisk denote the results of the propagation equation and the direct numerical integral, respectively. It is noted that the calculation results have some deviation near the edge of the aperture between the propagation equation and the direct numerical integral. The calculation error is 1.32%. The calculation time is saved greatly, and the ratio of corresponding computing time is 1:111 for the propagation equation and direct numerical integral.

It is noted that there is a deviation between series expansion and the transmittance function. To make the deviation

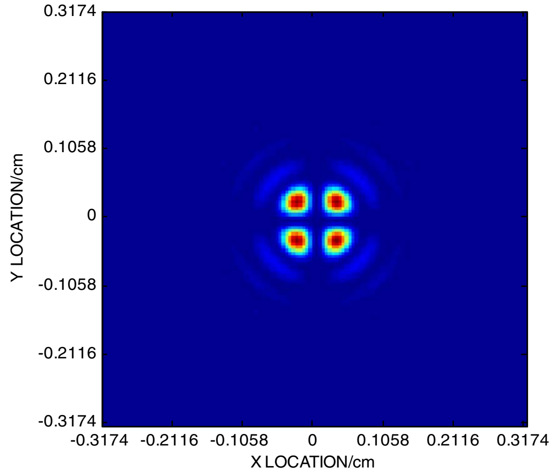

Fig. 4. Intensity distribution of TEM_{11}-mode Hermite–Gauss beam at the actual focus.

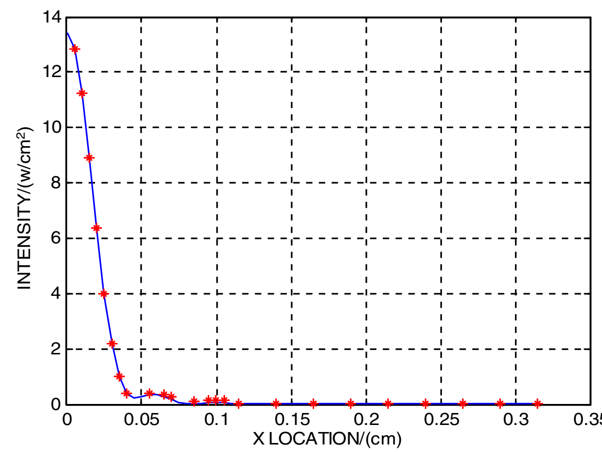

Fig. 5. Intensity distribution of TEM_{00}-mode Hermite–Gauss beam for both methods.

small enough to achieve the desired accuracy, we need to increase the numbers M and N of the base functions. When the numbers M and N are increased to be 20 and the other calculation parameters are the same, the calculation error is 0.93%. And the ratio of computing time is 1:145 for the expansion method and the direct numerical integral. In fact, we can make more calculations; however, the relevant discussion will be provided in the next paper for the sake of space limitations.

5. CONCLUSION

In this paper, the propagation equation of Hermite–Gauss beams through a complex optical system with limiting apertures is derived. The analytical propagation equation obtained in this paper is exact within the framework of paraxial approximation, and it is time saving compared with the numerical integration.

To obtain the analytical expression, we expand the aperture transmittance function into a finite sum of complex Gaussian functions. Thus, the limiting aperture is expressed as a superposition of a series of Gaussian-shaped limiting apertures. The advantage of this treatment is that we can handle almost all kinds of apertures by finding the set of coefficients u_k, v_l, ε_k, and φ_l in Eq. (16). It is noted that there is a deviation between series expansion and the transmittance function. To make the deviation small enough to achieve the desired accuracy, we need to increase the numbers M and N of the base functions. And the numbers M and N strongly depend on the transmittance function T [6]. For the sake of space limitations, the relevant discussion will be done in the next paper.

Owing to space limitations, we give only the approximate analytical expression of the beam width of the TEM_{11}-mode Hermite–Gauss beam in Section 4. In fact, we can generalize the results to the other modes by repeating the similar calculation with the aid of Eqs. (52) and (53) [8] as follows:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t \qquad (z > 0), \tag{52}$$

$$\gamma(a,x) = \int_0^x e^{-t} t^{a-1} \mathrm{d}t \qquad (a > 0), \tag{53}$$

where $\Gamma(z)$ is the gamma function and $\gamma(a,x)$ is the incomplete gamma function.

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