Linear equations method for modal decomposition using intensity information

Yuanyang Li,[1,2,*] Jin Guo,[1] Lisheng Liu,[1,2] Tingfeng Wang,[1] and Junfeng Shao[1,2]

[1]*State Key Laboratory of Laser Interaction with Matter, Changchun Institute of Optics, Fine Mechanics and Physics, Chinese Academy of Sciences, Changchun, Jilin 130033, China*
[2]*University of Chinese Academy of Sciences, Beijing 100039, China*
**Corresponding author: liyuanyang1108@sina.com*

Received September 5, 2013; revised October 18, 2013; accepted October 18, 2013;
posted October 18, 2013 (Doc. ID 197009); published November 11, 2013

The linear equations method is proposed to calculate the complete modal content of the partially coherent laser beam using only the intensity information. This method could give not only the incoherent expansion coefficients of the modal decomposition but also the cross-correlation expansion coefficients using the intensity profiles in several planes of finite distance along the propagation direction. A simulation is also presented to verify the validity of this theory. In our algorithm, the minimum and maximum mode orders should be known *a priori*, so we provide an estimation method for the two parameters.

OCIS codes: (030.4070) Modes; (030.1640) Coherence; (140.3295) Laser beam characterization; (140.3430) Laser theory; (140.3460) Lasers.
http://dx.doi.org/10.1364/JOSAA.30.002502

1. INTRODUCTION

It is very important to determine the transverse modal content of a laser beam in the beam characterization [1–6]. This work is closely related to the coherence property of the laser beam, which cannot be measured directly. The parameters of the laser beam, which can be measured directly, are not very abundant, and the intensity distribution is the easiest obtained information. Thus, getting the modal spectrum of the laser beam using only the intensity information becomes an essential issue in the problem of modal decomposition.

Gori *et al.* [7] first provide a proof about whether the intensity distribution is enough for a complete modal decomposition for a laser beam, and give a conclusion that the modal content can be determined for a one-dimensional case, but the result of a two-dimensional case may not be unique due to the existence of the vortex phase. However, this ambiguity in two-dimensional cases can be eliminated if we impose some limitation on the beam. For instance, we could give a limitation that there is no vortex phase in the beam, and in this situation, the modal content is accessible using only intensity information. Dragoman [8] even proves that the unambiguous coherence status of a vortex beam is available from intensity measurement alone with the help of an anamorphic optical system.

Based on previous work, Santarsiero *et al.* [9] give an innovative method for modal decomposition. With this method, one time measurement of the transverse intensity of the laser beam is sufficient for obtaining the modal content. But the insufficiency of the method is that it is only available for the beam that is made up of an incoherent mixture of Hermite–Gauss (HG) modes. The method of Santarsiero *et al.* is generalized by Xue *et al.* [10] for the beam that has a coherent mixture of HG modes applying several intensity profiles along the propagation direction. The generalized method of Xue *et al.* could give the incoherent expansion coefficients of modal expansion, but it cannot give the cross-correlation expansion coefficients. The other cross-correlation expansion coefficients are given by Laabs *et al.* [11], integrating the product of the ambiguity function and the Laguerre–Gauss function in phase space. And then based on the works of Xue and Laabs, Borghi *et al.* [12] point out that the complete modal structure can be recovered without using the ambiguity function. The method of Borghi *et al.* uses an integral over the whole space (from negative infinity to positive infinity) along the beam propagation direction. However, the application of this method to the practical problem is difficult due to the requirement of the intensity information in the whole space, which cannot be obtained from experimental measurements. We present a new method using the linear equations to give a complete modal decomposition result based on the previous work in [12]. Applying this method, we give the result of modal decomposition using the intensity information from several measurements in finite distances along the propagation direction, and overcome the difficulty of the integral over the whole space.

This paper is organized as follows. In Section 2, we give the basic theory of the linear equations method for modal decomposition. Section 3 mainly reports the simulation of this method in order to prove the correctness of the theory. The minimum and maximum mode orders should be known *a priori* for our modal decomposition algorithm, so we provide an estimation method to obtain the two parameters in Section 4. Finally, Section 5 presents our conclusion.

2. THEORETICAL ANALYSIS

A. Preliminaries

We only consider the one-dimensional case for simplicity. In the two-dimensional case, if the beam contains no vortex phase structure, the intensity information can determine the modal contents unambiguously [13]. Thus, extension to the two-dimensional case is straightforward for the vortex-free beam. Our method does not work when vortex phase structures are present in the modes of the source. However, other unambiguous analysis methods, for example, the method using computer-generated hologram-based correlation filters [14], exist when the beam is not vortex-free. We assume that the spatial scale, e.g., the waist width v_0 of the fundamental mode, can be derived from knowledge of the parameters of the laser cavity. For a partially coherent beam, the field distribution is considered as the superstition of a set of HG modes that all have a common waist position [1]. So we can express the field $U(x)$ as

$$U(x,\varphi) = \sqrt{\cos\varphi}\,\exp(ikz)\exp\left(\frac{ik}{2R}x^2\right) \times \sum_{n=0}^{\infty} c_n G_n(x\cos\varphi)\exp(-in\varphi), \tag{1}$$

where c_n are the modal coefficients and n denotes the mode number, and $k = 2\pi/\lambda$ is the wave number while z is the propagation distance. The parameter φ is the Gouy phase, which has the form of $\varphi = \tan^{-1}(z/f)$, where $f = kv_0^2/2$ is the Rayleigh distance. The Gouy phase φ changes from $-\pi/2$ to $\pi/2$ when the beam propagates from negative infinity to positive infinity. Equation (1) becomes more easily manipulated using φ to take the place of the propagation distance z. The parameter R denotes the radius of the wave front of the HG mode, which is useless to our discussion. The function $G_n(x)$ is the field distribution of the nth HG mode when the waist position is located at the transverse plane of $z = 0$. $G_n(x)$ is the form of

$$G_n(x) = \left(\frac{2}{\pi v_0^2}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\frac{x\sqrt{2}}{v_0}\right)\exp\left(-\frac{x^2}{v_0^2}\right), \tag{2}$$

where $H_n(x)$ are the nth-order Hermite polynomials.

The intensity distribution of the laser beam can be derived from Eqs. (1) and (2) as [12]

$$\begin{aligned} I(x,\varphi) &= \langle U(x,\varphi)U^*(x,\varphi)\rangle \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{2\cos\varphi}{1+\delta_{m,0}} \mathrm{Re}\{\langle c_n^* c_{n+m}\rangle \exp(-im\varphi)\} \\ &\quad \times G_n(x\cos\varphi)G_{n+m}(x\cos\varphi), \end{aligned} \tag{3}$$

where Re{} means selecting the real part of the function in the bracket and $\delta_{m,0}$ is the Kronecker function. In order to simplify the expression of Eq. (3), we introduce the coordinate scaled intensity distribution $\hat{I}(\xi,\varphi)$ by changing the variables as $\hat{I}(x,\varphi) = I(x,\varphi)/\cos\varphi$ and $\xi = x\cos\varphi$. Substituting this change into Eq. (3), we get that

$$\hat{I}(\xi,\varphi) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{2}{1+\delta_{m,0}} \mathrm{Re}\{\langle c_n^* c_{n+m}\rangle \exp(-im\varphi)\} \times G_n(\xi)G_{n+m}(\xi). \tag{4}$$

The coordinate scaled intensity distribution $\hat{I}(\xi,\varphi)$ disregards the attenuation and transverse expansion of $I(x,\varphi)$, which are caused by the propagation of the laser beam, because they have no connection with the mode superposition effect.

B. Linear Equations Method for Modal Decomposition

The expansion coefficients $\langle c_n^* c_{n+m}\rangle$ in Eq. (4) are generally complex numbers, so we can express them as

$$\langle c_n^* c_{n+m}\rangle = M_{n,m}\exp(i\theta_{n,m}). \tag{5}$$

$M_{n,m}$ and $\theta_{n,m}$ are the amplitudes and arguments of the expansion coefficients $\langle c_n^* c_{n+m}\rangle$, respectively. The amplitudes $M_{n,m}$ represent the power contents of the correlated modes, and the arguments $\theta_{n,m}$ are the phase delays between the transverse modes of order n and $n+m$. The practical optical beam generally has finite modes [10], so we can define a lowest-order n_0 and a highest-order N for modal coefficients c_n in Eq. (1). Substituting Eq. (5) into Eq. (4) and considering that the mode order is limited, we can obtain

$$\hat{I}(\xi,\varphi) = \sum_{m=0}^{N-n_0}\sum_{n=n_0}^{N-m} \frac{2}{1+\delta_{m,0}} M_{n,m}\cos(m\varphi - \theta_{n,m}) \times G_n(\xi)G_{n+m}(\xi). \tag{6}$$

The maximum value of m in Eq. (6) is $N - n_0$. If m is determined, the mode order n is limited in the range of n_0 to $N - m$ and the expansion coefficients $\langle c_n^* c_{n+m}\rangle$ are zeros out of this range. We introduce two new parameters $A_m(x)$ and $B_m(x)$ here for further discussion:

$$A_m(\xi) = 2\sum_{n=n_0}^{N-m} \frac{1}{1+\delta_{m,0}} M_{n,m}\cos\theta_{n,m} G_n(\xi)G_{n+m}(\xi), \tag{7}$$

$$B_m(\xi) = 2\sum_{n=n_0}^{N-m} M_{n,m}\sin\theta_{n,m} G_n(\xi)G_{n+m}(\xi). \tag{8}$$

Taking Eqs. (7) and (8) into Eq. (6), we can change Eq. (6) into a series form:

$$\hat{I}(\xi,\varphi) = \sum_{m=0}^{N-n_0} [A_m(\xi)\cos(m\varphi) + B_m(\xi)\sin(m\varphi)]. \tag{9}$$

The intensity information for modal decomposition can be given from the experiments. Considering the experimental conditions in practice, some processes are needed to solve the expansion coefficients $\langle c_n^* c_{n+m}\rangle$. First, one can only measure the intensity distribution at a finite number of transverse planes in finite distances. If we measure T times along the propagation direction, the intensity distribution function $\hat{I}(\xi,\varphi)$ only has certain values at the planes of $\varphi = \varphi_1, \varphi_2, \ldots, \varphi_T$. Second, one must use a kind of sensor, a CCD, for example, to measure the intensity. The sensors sample the transverse coordinate ξ discretely. We can define the sampling number in the transverse coordinate as S. Based on the two procedures given above, the intensity distribution function $\hat{I}(\xi,\varphi)$ changes to a matrix of order $T \times S$. Furthermore, Eq. (9) becomes linear equations due to the discrete sampling of φ, and this will be shown more clearly in the following discussion.

In order to modify Eq. (9) into matrix form, we introduce the matrices below:

$$\mathbf{P}_1 = \begin{pmatrix} \cos(0\varphi_1) & \dots & \cos[(N-n_0)\varphi_1] \\ \vdots & \ddots & \vdots \\ \cos(0\varphi_T) & \cdots & \cos[(N-n_0)\varphi_T] \end{pmatrix},$$

$$\mathbf{P}_2 = \begin{pmatrix} \sin(0\varphi_1) & \dots & \sin[(N-n_0)\varphi_1] \\ \vdots & \ddots & \vdots \\ \sin(0\varphi_T) & \cdots & \sin[(N-n_0)\varphi_T] \end{pmatrix}; \quad (10)$$

$$\mathbf{I} = \begin{pmatrix} I(\xi_1,\varphi_1) & \dots & I(\xi_S,\varphi_1) \\ \vdots & \ddots & \vdots \\ I(\xi_1,\varphi_T) & \cdots & I(\xi_S,\varphi_T) \end{pmatrix}; \quad (11)$$

$$\mathbf{A} = (\mathbf{A}_0, \mathbf{A}_1, \quad \dots \quad \mathbf{A}_{N-n_0})^T,$$
$$\mathbf{B} = (\mathbf{B}_0, \mathbf{B}_1, \quad \dots \quad \mathbf{B}_{N-n_0})^T, \quad (12)$$

where

$$\mathbf{A}_m = (A_m(\xi_1) \quad \dots \quad A_m(\xi_S)),$$
$$\mathbf{B}_m = (B_m(\xi_1) \quad \dots \quad B_m(\xi_S)). \quad (13)$$

The elements of matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ are the functions of the Gouy phase φ. If the sampling status of φ is determined, $\mathbf{P}_1$ and $\mathbf{P}_2$ can be treated as known matrices. The intensity distribution matrix $\mathbf{I}$ is obtained from experimental measurements, whose rows are the transverse intensity profiles at each plane and columns are the intensity distributions along the propagation direction. $\mathbf{A}$ and $\mathbf{B}$ are unknown matrices, and the solving method for these two matrices will be provided below. Substituting all the matrices into Eq. (9), we can represent Eq. (9) in matrix form as

$$\mathbf{I} = \mathbf{P}_1\mathbf{A} + \mathbf{P}_2\mathbf{B} = (\mathbf{P}_1 \quad \mathbf{P}_2)\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}. \quad (14)$$

Let $\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2)$ and $\Omega = (\mathbf{A}, \mathbf{B})^T$; Eq. (14) is rewritten as

$$\mathbf{P}\Omega = \mathbf{I}. \quad (15)$$

Equation (15) is a typical form of a set of linear equations. The linear equations are solved by

$$\Omega = \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} = \mathbf{P}^{+}\mathbf{I}, \quad (16)$$

where $\mathbf{P}^{+}$ denotes the Moore–Penrose pseudo-inverse matrix of $\mathbf{P}$. If the intensity distribution in terms of φ at a certain transverse coordinate ξ is treated as a signal, the maximum signal frequency is $f_m = (N-n_0)/(2\pi)$ according to Eq. (9). The sampling frequency of the signal is $f_s = T/\pi$, where T is the sampling number of φ as defined before. In order to prevent the aliasing error, the Nyquist sampling theorem [15] suggests that $f_s \geq 2f_m$. After some rearrangement, the inequality becomes $T \geq N - n_0$. If we choose the value of T satisfying $T > N - n_0$ in order to get a robust result, the set of linear equations in Eq. (15) is inconsistent. Based on the theory of the pseudo-inverse matrix, Ω is the least squared solution of Eq. (15) with the minimum norm, and it is unique [16]. The solution Ω is a matrix that has an order of $2(N-n_0+1) \times S$ according to Eq. (16).

Equations (7) and (8) are the functions of the scaled coordinate ξ whose value is also discrete; therefore they can be transformed into two sets of linear equations like Eq. (9). Some matrices are defined for this transformation:

$$\mathbf{G}_m = \begin{pmatrix} G_{n_0}(\xi_1)G_{n_0+m}(\xi_1) & \cdots & G_{N-m}(\xi_1)G_N(\xi_1) \\ \vdots & \ddots & \vdots \\ G_{n_0}(\xi_S)G_{n_0+m}(\xi_S) & \cdots & G_{N-m}(\xi_S)G_N(\xi_S) \end{pmatrix}, \quad (17)$$

$$\mathbf{C}_{\mathrm{real},m} = (M_{n_0,m}\cos\theta_{n_0,m} \quad \dots \quad M_{N-m,m}\cos\theta_{N-m,m})^T,$$
$$\mathbf{C}_{\mathrm{imag},m} = (M_{n_0,m}\sin\theta_{n_0,m} \quad \dots \quad M_{N-m,m}\sin\theta_{N-m,m})^T. \quad (18)$$

$\mathbf{G}_m$ is the same as $\mathbf{P}_1$ and $\mathbf{P}_2$, which is a known matrix if the sampling value of ξ is determined. The elements of the two vectors of Eq. (18) are unknown quantities. Substituting Eqs. (13), (17), and (18) into Eqs. (7) and (8), we have

$$\mathbf{G}_m\mathbf{C}_{\mathrm{real},m} = \frac{1+\delta_{m,0}}{2}\mathbf{A}_m^T, \quad (19)$$

$$\mathbf{G}_m\mathbf{C}_{\mathrm{imag},m} = \mathbf{B}_m^T. \quad (20)$$

Equations (19) and (20) are two sets of linear equations for a particular value of m. The sampling rate of the intensity sensor is much higher than the mode order, so we have the inequality of $S \gg N - m - n_0 + 1$. This means that Eqs. (19) and (20) are both inconsistent and their solutions can be obtained using the same method as Eq. (16). Therefore, the solutions of Eqs. (19) and (20) are

$$\mathbf{C}_{\mathrm{real},m} = \frac{1+\delta_{m,0}}{2}\mathbf{G}_m^{+}\mathbf{A}_m^T, \quad (21)$$

$$\mathbf{C}_{\mathrm{imag},m} = \mathbf{G}_m^{+}\mathbf{B}_m^T, \quad (22)$$

where $\mathbf{G}_m^{+}$ is the Moore–Penrose pseudo-inverse matrix of $\mathbf{G}_m$. $\mathbf{A}_m$ and $\mathbf{B}_m$ are the row vectors of $\mathbf{A}$ and $\mathbf{B}$, which can be derived from Eq. (16). Given a certain value of m, the vectors $\mathbf{C}_{\mathrm{real},m}$ and $\mathbf{C}_{\mathrm{imag},m}$ are the solutions of Eqs. (19) and (20), respectively, and they can be solved by the multiplication of matrices of $\mathbf{G}_m^{+}$, $\mathbf{A}_m$, and $\mathbf{B}_m$ using Eqs. (21) and (22). Notice that the elements of vectors $\mathbf{C}_{\mathrm{real},m}$ and $\mathbf{C}_{\mathrm{imag},m}$ are the real part and image part of the expansion coefficients $\langle c_n^* c_{n+m}\rangle$; thus the expansion coefficients can be derived as

$$\mathbf{C}_m = (\langle c_{n_0}^* c_{n_0+m}\rangle \quad \dots \quad \langle c_{n_0+N-m}^* c_{n_0+N}\rangle)^T = \mathbf{C}_{\mathrm{real},m} + i\mathbf{C}_{\mathrm{imag},m}. \quad (23)$$

Based on the discussions given above, we can synthesize the processes for solving the expansion coefficients $\langle c_n^* c_{n+m}\rangle$ into five steps:

1. First, we need to obtain the transverse intensity profiles at several propagation distances. The intensity profile of the laser beam is measured using a sensor such as a CCD. If we measure the intensity T times, it must satisfy the relation of $T > N - n_0$ due to the Nyquist theorem as we mentioned before. The minimum and maximum values of the mode number n are n_0 and N, respectively. They must be known *a priori* in our algorithm, and the method for estimating them is given in detail in Section 4.
2. Using the intensity matrix $\mathbf{I}$ in step one and Eq. (16), we have a matrix Ω as a result.
3. Matrix Ω has $2(N - n_0 + 1)$ rows according to Eqs. (12) and (16). We need to choose two rows, $\mathbf{A}_m$ and $\mathbf{B}_m$, from matrix Ω and substitute them into Eqs. (21) and (22), respectively. The results are the solutions of vectors $\mathbf{C}_{\mathrm{real},m}$ and $\mathbf{C}_{\mathrm{imag},m}$.
4. We obtain a set of expansion coefficients $\mathbf{C}_m$ after substituting $\mathbf{C}_{\mathrm{real},m}$ and $\mathbf{C}_{\mathrm{imag},m}$ into Eq. (23).
5. Using steps 3 and 4, we can only get the expansion coefficients when m has a certain value. In order to obtain the complete sets of the expansion coefficients, we should repeat steps 3–4 until all the values of m from 0 to $N - n_0$ have been calculated. Finally, we can get the complete sets of expansion coefficients $\langle c_n^* c_{n+m}\rangle$, which are the modal structure of the beam.

After obtaining the complete sets of expansion coefficients by following the procedures given above, we can use them to determine the modal coefficients c_n if the modes are superposed coherently. The amplitudes and arguments of c_n are computed separately. At first, the amplitudes of c_n are given by

$$|c_n| = \sqrt{\langle c_n^* c_n\rangle}. \tag{24}$$

When the mode number n satisfies the inequality $n_0 < n < N$, the arguments of c_n can be solved by

$$\arg(c_n) = \frac{1}{2}[\arg(\langle c_{n-1}^* c_{n+1}\rangle) - \arg(\langle c_{n-1}^* c_n\rangle) - \arg(\langle c_n^* c_{n+1}\rangle)], \tag{25}$$

where $\arg(c_n)$ means the arguments of c_n. After solving $\arg(c_{n0+1})$ and $\arg(c_{N-1})$ using Eq. (25), the arguments of the order n_0 and N, which are the boundary values of n, are given using the two simple formulas

$$\begin{aligned}\arg(c_{n_0}) &= -\arg(\langle c_{n_0}^* c_{n_0+1}\rangle) - \arg(c_{n_0+1}),\\ \arg(c_N) &= -\arg(\langle c_{N-1}^* c_N\rangle) - \arg(c_{N-1}).\end{aligned} \tag{26}$$

Finally, the transverse mode structure of the laser beam is determined completely after solving all the modal coefficients c_n.

The discussions given above are based on the assumption that all the modes are superposed coherently. However, we want to state that our method is still available for the incoherent superposition of the modes. We suppose that there exists a beam composed of two HG modes $G_i(x)$ and $G_j(x)$ that are completely incoherent to each other, and the their modal coefficients are c_i and c_j. Taking the coefficients into Eq. (3), we have the intensity distribution of the beam as

$$I(x) = |c_i|^2 G_i^2(x) + |c_j|^2 G_j^2(x) + \mathrm{Re}\{\langle c_i^* c_j\rangle\} G_i(x) G_j(x). \tag{27}$$

Equation (3) can be proved correct only if the value of $\langle c_i^* c_j\rangle$ is zero in this incoherent situation in Eq. (27). In fact, the value of $\langle c_i^* c_j\rangle$ is zero indeed as we expect because c_i and c_j have no certain phase relation and cannot interfere in a stationary way, which results in the average of $c_i^* c_j$ being zero. Thus, Eq. (27) becomes the sum of the intensity distributions of the two beams, which proves the correctness of Eq. (3). Thus Eq. (3) is still available for the incoherent modal superposition though it is derived from Eq. (1), which cannot describe the incoherent beams. Thanks to the fact that the derivation of this modal decomposition algorithm is based on Eq. (3), we can state that our method is correct for the cases in which the beams are coherent, incoherent, or a mixture superposition of the modes, and the processes of the algorithm are all the same for these cases. However, for the cases in which the incoherent superpositions exist, the product $c_n^* c_{n+m}$ does not equal the value of $\langle c_n^* c_{n+m}\rangle$. Thus, we should not describe the beams using the modal coefficients c_n but using the expansion coefficients $\langle c_n^* c_{n+m}\rangle$ directly.

3. SIMULATION

In order to verify the correctness of our theory, we use Eq. (1) to construct a virtual beam that is a coherent superposition of the modes. The modal coefficients c_n need to be chosen randomly in order to ensure the generality of the modal decomposition result. Table 1 shows a set of modal coefficients that are randomly chosen. The amplitudes and arguments in Table 1 are given separately because c_n are complex values. The values of c_n that are not included in Table 1 are set to be zeros. Substituting the coefficients into Eq. (4) and setting the waist width v_0 as 1 mm, we have the coordinate scaled intensity distribution $\hat{I}(\xi, \varphi)$ of this virtual beam and the intensity profiles of the beam at several planes are shown in Fig. 1. The intensity profile is varied when the optical beam propagates, which is due to the existence of the cross-correlation terms of the expansion coefficients $\langle c_n^* c_{n+m}\rangle$. If the modal structure of the beam is composed of the incoherent mixture of the modes, which means that the cross-correlation terms of $\langle c_n^* c_{n+m}\rangle$ are zeros, the intensity distribution is not varied with respect to the propagation distance z or the Gouy phase φ.

We need to sample the intensity distribution before the modal decomposition. First, the beam intensity is measured at 20 different positions along the propagation direction, and the positions are determined by sampling the Gouy phase φ at an even space between $-2/5\pi$ and $2/5\pi$ for simplicity. With this sampling range of φ, we can find that all the sampling points are located between the propagation distances $z = -3f$ and $3f$ where f is the Rayleigh distance. Second, the intensity distribution is sampled at each transverse plane using 1000 evenly spaced points between the transverse coordinates $\xi = -5v_0$ and $\xi = 5v_0$. According to Table 1, the minimum mode order n_0 is 3 and the maximum one N is 7. The results

Table 1. Mode Coefficients c_n of the nth HG Mode

	c_3	c_4	c_5	c_6	c_7
Amplitude	0.3	0.1	0.2	0.2	0.2
Argument	$3/10\pi$	$-1/2\pi$	$2/5\pi$	$-3/7\pi$	0

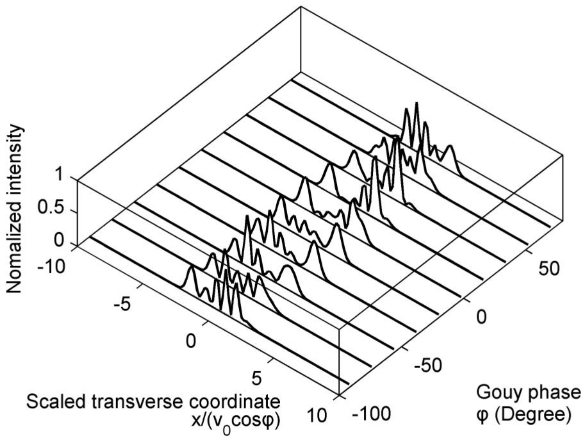

Fig. 1. Intensity profiles of the beam along the propagation direction.

of modal decomposition are shown in Figs. 2 and 3 after substituting all the information above into our algorithm.

Figure 2 shows the simulations and theoretical results of the expansion coefficients $\langle c_n^* c_{n+m} \rangle$ when $m = 0$, 1, 2, and 3. If $m = 0$, the expansion coefficients $\langle c_n^* c_{n+m} \rangle$ represent the power contents of the modes. And if $m > 0$, the expansion coefficients determine the phase relation of the modes. Due to the coherent superposition of the modes in this case, we have the relation that $\langle c_n^* c_{n+m} \rangle = c_n^* c_{n+m}$. Comparing the results in Fig. 2, we hardly find any difference between the results of our algorithm and the values computed by Table 1. This confirms that our algorithm could give a very exact result of the amplitude of the expansion coefficients $\langle c_n^* c_{n+m} \rangle$. Figure 3 shows the arguments of the corresponding expansion coefficients in Fig. 2. The arguments of the expansion coefficients $\langle c_n^* c_n \rangle$ must be zero because $\langle c_n^* c_n \rangle$ are real values. In Fig. 3(a), the results of the algorithm are consistent with this fact. Figure 3(b) shows the results of the algorithm and the strict solutions when $m = 1$. We can see that the simulation error is also very small in this figure. However, when m has a value of 2 or 3, some results of this algorithm are not the same as what we expect. Taking the case of $m = 2$ for example, the argument of $\langle c_6^* c_8 \rangle$ should be $3\pi/7$, which can be derived from Table 1, but the algorithm gives a zero value as a result. In fact, the amplitude of $\langle c_6^* c_8 \rangle$ is zero according to Fig. 2(c), which means that the argument of $\langle c_6^* c_8 \rangle$ is not important in this situation, because a complex number remains zero when its amplitude is zero, no matter what its argument is. This means that our algorithm is accurate enough for solving the arguments of the expansion coefficients $\langle c_n^* c_{n+m} \rangle$. Using the results in this simulation, the mode coefficients c_n can be obtained by Eqs. (24), (25), and (26) without effort.

The correctness of the linear equations method described in Section 2.B is confirmed by this simulation. However, without knowing the minimum mode order n_0 and maximum one N, this simulation cannot come into existence for practical laser beams. In fact, we can set the parameter n_0 to be zero for practical consideration, because low-order modes always exist for a real laser beam unless some special techniques,

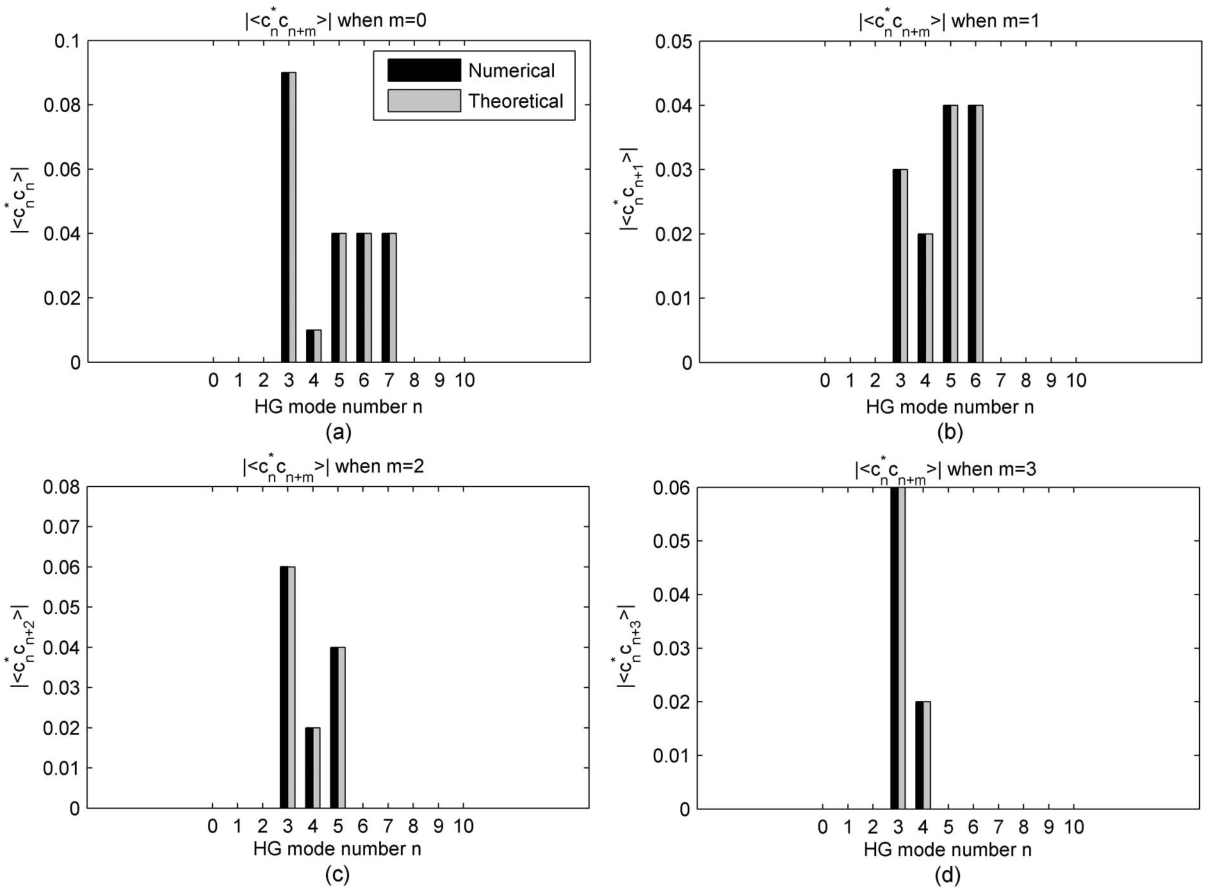

Fig. 2. Theoretical and numerical results of the amplitudes of the expansion coefficients $\langle c_n^* c_{n+m} \rangle$.

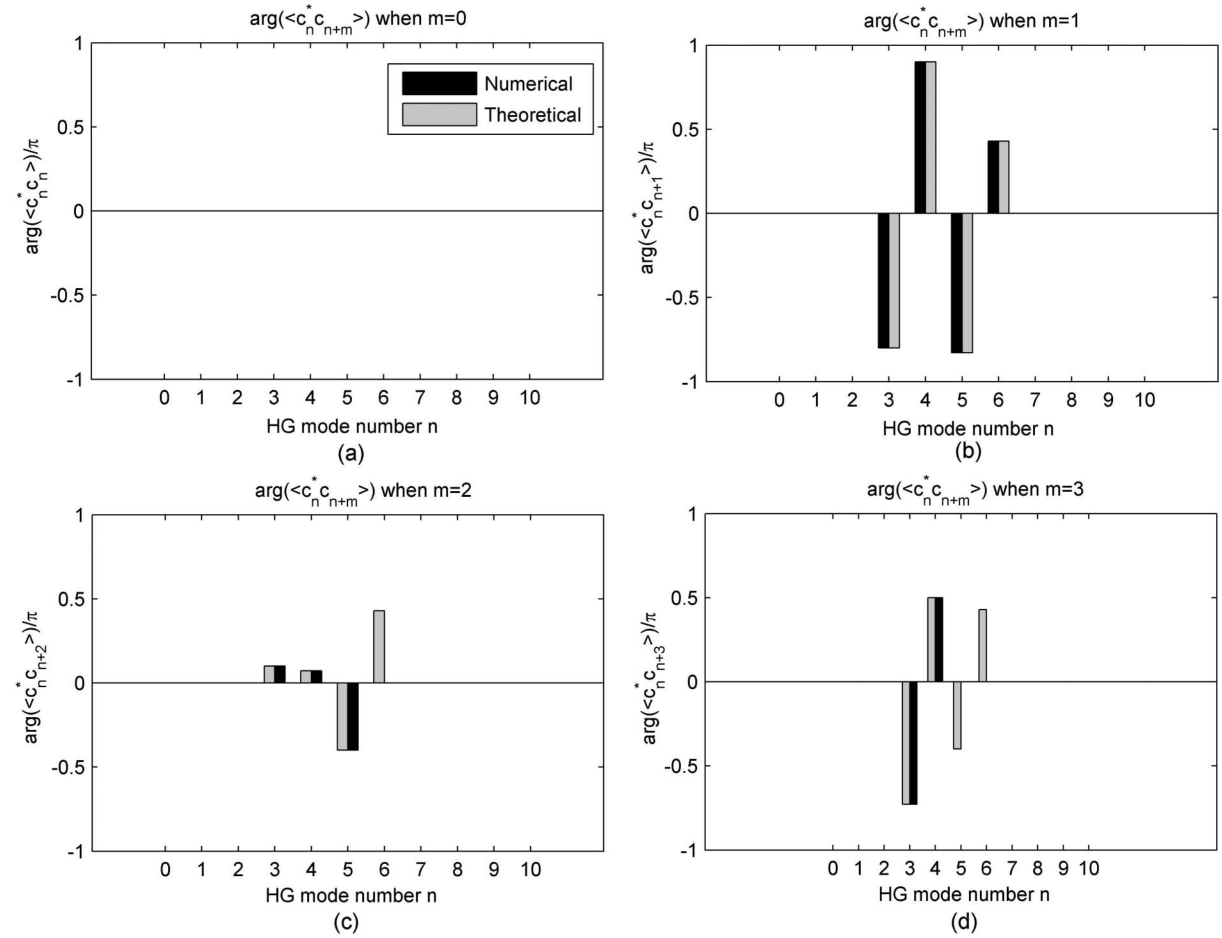

Fig. 3. Theoretical and numerical results of the arguments of the expansion coefficients $\langle c_n^* c_{n+m}\rangle$.

using an intracavity amplitude mask, for example, are applied to the laser source. But things are different for the parameter N. If the chosen value of N is too small, we cannot obtain a correct result. If the value of N is given too large, this could cause great calculation burden. Thus giving a method to find the value of N is necessary.

4. HOW TO FIND THE MAXIMUM MODE ORDER N

In this section, we provide a method to estimate the parameter N, which cannot be measured directly. A method that is used to prove that N is not infinity for a real beam has been proposed in [10]. But we cannot use this method to give an estimation of N. The intensity distribution $\hat{I}(\xi,\varphi)$ can be represented as a finite Fourier series of φ; therefore the cutoff frequency of the Fourier transform of $\hat{I}(\xi,\varphi)$ should be $(N-n_0)/(2\pi)$ based on Eq. (9). This supposes to be a good beginning to estimate N if the cutoff frequency can be calculated accurately. But we can never give a cutoff frequency accurate enough unless we can sample the function $\hat{I}(\xi,\varphi)$ of φ in its 2π period. It is impossible to measure $\hat{I}(\xi,\varphi)$ in terms of φ from $-\pi$ to π because the Gouy phase is limited in the range of $-\pi/2$ to $+\pi/2$ and the values out of this range are meaningless in the physical sense. So the cutoff frequency method for estimating N is not appropriate as well. We propose a new method to estimate N using the average of the intensities along the propagation direction.

We measure the intensity distributions in the range of $-\varphi_0$ to φ_0 of the Gouy phase φ with even space along the beam propagation direction. We denote the total sampling numbers as K and the sampling interval as $\Delta\varphi = 2\varphi_0/(K-1)$. Thus, the positions where the intensity distribution is measured can be presented as $\varphi_k = -\varphi_0 + k\Delta\varphi$, $k = 1, 2, 3\ldots$ The average of these measured intensity data along the propagation direction is then given by

$$\begin{aligned}\langle \hat{I}(\xi)\rangle_\varphi &= A_0(\xi) + \frac{1}{K}\sum_{m=1}^{N-n_0}\sum_{k=1}^{K}[A_m(\xi)\cos(m\varphi_k) + B_m(\xi)\sin(m\varphi_k)]\\ &= A_0(\xi) + \frac{1}{K}\sum_{m=1}^{N-n_0}\sum_{k=1}^{K}A_m(\xi)\cos(m\varphi_k). \qquad (28)\end{aligned}$$

The fact that sine is an odd function is used in Eq. (28). It can be noted that $A_m(\xi)$ is an even function when m is an even number and it is an odd function when m is odd. Taking the parity of $A_m(\xi)$ into consideration, we have

$$\begin{aligned}\tilde{A}_0(\xi) &= \frac{\langle \hat{I}(\xi)\rangle_\varphi + \langle \hat{I}(-\xi)\rangle_\varphi}{2}\\ &= A_0(\xi) + \frac{1}{K}\sum_{h=1}^{\left[\frac{N-n_0}{2}\right]} A_{2h}(\xi)\sum_{k=1}^{K}\cos(m\varphi_k), \qquad (29)\end{aligned}$$

where $[(N-n_0)/2]$ means to take the largest integral number that is smaller than $(N-n_0)/2$. The function $\tilde{A}_0(\xi)$ can be treated as an estimation of $A_0(\xi)$ and the error of this estimation is determined by the last term in Eq. (29). There are only even order terms of $A_m(\xi)$ in Eq. (29), so we need to discuss the these terms in detail below.

According to Eq. (7), the function $A_{2h}(\xi)$ can be expanded by the bases $G_n(\xi)G_{n+2h}(\xi)$, which we can denote as

$$A_{2h}(\xi)=\mathrm{Span}\{G_0(\xi)G_{2h}(\xi),G_1(\xi)G_{2h+1}(\xi),\ldots,G_{N-2h}(\xi)G_N(\xi)\}. \tag{30}$$

The bases $G_n(\xi)G_{n+2h}(\xi)$ are unrelated to each other. The Fourier transform of the bases can be represented in a closed form as [12]

$$\mathcal{F}\{G_n(\xi)G_{n+m}(\xi)\}(u)=(-i)^m\psi_n^m(\pi^2v_0^2u^2), \tag{31}$$

where

$$\psi_n^m(x)=R_nx^{\frac{m}{2}}L_n^m(x)\exp\left(-\frac{x}{2}\right), \tag{32}$$

R_n are constants related with the order n, and L_n^m are the generalized Laguerre polynomials. Based on the recurrence relation of the generalized Laguerre polynomials [17], we can derive the following relation:

$$xL_n^{2h}(x)=(2h-1)\sum_{m=0}^{n}L_m^{2(h-1)}(x)-(n+1)L_{n+1}^{2(h-1)}(x). \tag{33}$$

Substituting Eq. (32) into Eq. (33) and then taking the inverse Fourier transform of Eq. (33), the bases $G_n(\xi)G_{n+2h}(\xi)$ are represented by another set of bases $G_n(\xi)G_{n+2(h-1)}(\xi)$ as

$$G_n(\xi)G_{n+2h}(\xi)=-R_n\left[\sum_{m=0}^{n}\frac{(2h-1)}{R_m}G_m(\xi)G_{m+2(h-1)}(\xi)-\frac{n+1}{R_{n+1}}G_{n+1}(\xi)G_{(n+1)+2(h-1)}(\xi)\right]. \tag{34}$$

Applying Eq. (34) to all of the bases in Eq. (30) recurrently, the bases used to expand $A_{2h}(\xi)$ are finally changed to be

$$A_{2h}(\xi)=\mathrm{Span}\{G_0(\xi)^2,G_1(\xi)^2,\ldots,G_{N-h}(\xi)^2\}. \tag{35}$$

The function $\tilde{A}_0(\xi)$ is considered as the sum of the functions $A_{2h}(\xi)$ according to Eq. (29), which means that $\tilde{A}_0(\xi)$ has all the bases of the functions $A_{2h}(\xi)$ when h is from 0 to $[(N-n_0)/2]$. This suggests that

$$\tilde{A}_0(\xi)=\mathrm{Span}\{G_0(\xi)^2,G_1(\xi)^2,\ldots,G_N(\xi)^2\}. \tag{36}$$

Equations (35) and (36) mean that the functions $A_0(\xi)$ and $\tilde{A}_0(\xi)$ have the same bases and we can obtain maximum mode order N from the expansion result of $\tilde{A}_0(\xi)$. After replacing $A_0(\xi)$ by $\tilde{A}_0(\xi)$ in our modal decomposition algorithm, the procedures to obtain the expansion coefficients $\langle c_n^*c_n\rangle$ can be employed to expand the function $\tilde{A}_0(\xi)$ using the bases $G_n(\xi)^2$, and the maximum expanding order is considered to be N. The estimation method of N is described in detail as follows:

1. First, the average of the evenly sampled intensity distribution $\hat{I}(\xi,\varphi)$ is calculated. Substituting the result into Eq. (29), we obtain the function $\tilde{A}_0(\xi)$.
2. Next we expand the function $\tilde{A}_0(\xi)$ by replacing the function $A_0(\xi)$ by $\tilde{A}_0(\xi)$ in Eq. (21). Before solving Eq. (21), we should determine the order of the matrix $\mathbf{G}_0$. Here we give zero value to n_0 and an arbitrary number of columns to the matrix $\mathbf{G}_0$ at first.
3. We can treat the maximum order of the nonzero terms of the expansion result as the real value of N. If the number of columns of $\mathbf{G}_0$ that we choose arbitrarily is too small, we cannot determine the maximum order of the nonzero terms. In this case, we should expand the number of columns of $\mathbf{G}_0$ and solve Eq. (21) again until we find the nonzero term of the maximum order.

The method given above is very similar to using our modal decomposition algorithm as a trial repeatedly to find the correct value of the maximum mode order N; however, we want to state that this estimation method has two advantages compared with using the modal decomposition algorithm directly. First, this estimation method does not have a demand that the sampling number K must satisfy the sampling criterion. Thus measuring the laser intensity three or four times is enough to perform this task. Second, the function $\tilde{A}_0(\xi)$ is obtained

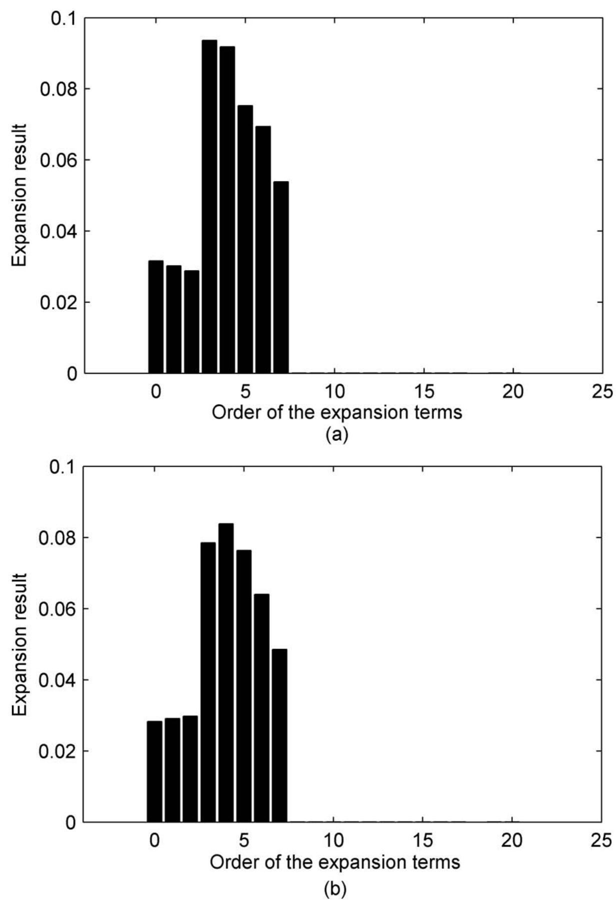

Fig. 4. Expansion results of the function $\tilde{A}_0(\xi)$ with different sampling numbers. (a) The sampling number K is 3. (b) The sampling number K is 5.

directly from the average of the sampling data, so we do not need to solve a set of linear equations as what we do to obtain $A_0(\xi)$. Based on the two advantages, we can greatly increase the speed to find the value of N using a very small amount of data and hardware resources.

We use the data gathered from the virtual beam constructed in Section 3 to examine the correctness of this estimation method. The expansion results of the function $\tilde{A}_0(\xi)$ using the bases $G_n(\xi)^2$ are shown in Fig. 4, where we use two different kinds of sampling information of intensity distributions to calculate the function $\tilde{A}_0(\xi)$. In Fig. 4(a), we set the value of φ_0 to be $\pi/5$, which defines a sampling range from $-\pi/5$ to $\pi/5$ for the Gouy phase, and the total sampling number K to be 3. The sampling range in Fig. 4(b) remains the same as that in Fig. 4(a), but the sampling number K is set to be 5. Comparing the two expansion results in Figs. 4(a) and 4(b), it is interesting to note that the maximum orders of the nonzero terms are both 7, which is the exact value of the maximum mode order indicated by Table 1, though the sampling numbers of the data to obtain $\tilde{A}_0(\xi)$ are different. Thus these results prove the correctness of the derivation for this estimation method and also indicate that this estimation method needs a lower sampling rate than the modal decomposition algorithm.

5. CONCLUSION

This paper introduces a modal decomposition method using the intensity information. The modal coefficients are obtained by solving the sets of linear equations in our theory. The required intensity information is discrete both in the propagation direction and in the transverse section of a plane since we use matrices in this algorithm. This discretization is of practical meaning because we cannot obtain a continuous intensity distribution from experimental measurements. During the calculation, we only use the intensity information, which is limited in finite distances and overcomes the difficulty of the integral in the whole space in [12].

The simulation for a virtual beam whose modal components are already known is taken and the results show that the solutions for arguments and amplitudes of the expansion coefficients $\langle c_n^* c_{n+m}\rangle$ are exactly consistent with the theoretical values. In our modal decomposition algorithm, the minimum mode order n_0 and the maximum mode order N should be known *a priori*. The value of n_0 is generally set to be zero for practical consideration. However, the parameter N is harder to find and an inappropriate value of N may cause great calculation burden. Thus we propose an estimation method to find the value of N quickly without any prior knowledge about the mode contents of the beam. A simulation proves that this estimation method could give the exact value of N and could lead to efficiencies for searching the value of N due to the lower sampling number and more simple procedures.

ACKNOWLEDGMENTS

This work was supported by the State Key Laboratory Foundation of China.

REFERENCES

1. A. E. Siegman, *Lasers* (University Science Books, 1986).
2. E. Wolf and G. S. Agarwal, “Coherence theory of laser resonator modes,” J. Opt. Soc. Am. A **1**, 541–546 (1984).
3. C. Schulze, D. Naidoo, D. Flamm, O. A. Schmidt, A. Forbes, and M. Duparre, “Wavefront reconstruction by modal decomposition,” Opt. Express **20**, 19714–19725 (2012).
4. O. A. Schmidt, C. Schulze, D. Flamm, R. Bruning, T. Kaiser, S. Schroter, and M. Duparre, “Real-time determination of laser beam quality by modal decomposition,” Opt. Express **19**, 6741–6748 (2011).
5. R. Borghi, G. Piquero, and M. Santarsiero, “Use of biorthogonal functions for the modal decomposition of multimode beams,” Opt. Commun. **194**, 235–242 (2001).
6. K. M. Du, G. Herziger, P. Loosen, and F. Ruhl, “Computation of the statistical properties of laser light,” Opt. Quantum Electron. **24**, S1095–S1108 (1992).
7. F. Gori, M. Santarsiero, and G. Guattari, “Coherence and the spatial distribution of intensity,” J. Opt. Soc. Am. A **10**, 673–679 (1993).
8. D. Dragoman, “Unambiguous coherence retrieval from intensity measurements,” J. Opt. Soc. Am. A **20**, 290–295 (2003).
9. M. Santarsiero, F. Gori, R. Borghi, and G. Guattari, “Evaluation of the modal structure of light beams composed of incoherent mixtures of Hermite-Gaussian modes,” Appl. Opt. **38**, 5272–5281 (1999).
10. X. Xue, H. Wei, and A. G. Kirk, “Intensity-based modal decomposition of optical beams in terms of Hermite-Gaussian functions,” J. Opt. Soc. Am. A **17**, 1086–1091 (2000).
11. H. Laabs, B. Eppich, and H. Weber, “Modal decomposition of partially coherent beams using the ambiguity function,” J. Opt. Soc. Am. A **19**, 497–504 (2002).
12. R. Borghi, G. Guattari, L. de la Torre, F. Gori, and M. Santarsiero, “Evaluation of the spatial coherence of a light beam through transverse intensity measurements,” J. Opt. Soc. Am. A **20**, 1763–1770 (2003).
13. T. E. Gureyev, A. Roberts, and K. A. Nugent, “Partially coherent fields, the transport-of-intensity equation, and phase uniqueness,” J. Opt. Soc. Am. A **12**, 1942–1946 (1995).
14. T. Kaiser, D. Flamm, S. Schroter, and M. Duparr, “Complete modal decomposition for optical fibers using CGH-based correlation filters,” Opt. Express **17**, 9347–9356 (2009).
15. J. D. Schmidt, *Numerical Simulation of Optical Wave Propagation with Examples in MATLAB* (SPIE, 2010).
16. H. Yanai, K. Takeuchi, and Y. Takane, *Projection Matrices, Generalized Inverse Matrices, and Singular Value Decomposition* (Springer, 2011).
17. G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicist* (Academic, 2013).