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INDUSTRIAL APPLICATION

Design multiple-layer gradient coils using least-squares finite element method

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Abstract The design of gradient coils for magnetic resonance imaging is an optimization task in which a specified distribution of the magnetic field inside a region of interest is generated by choosing an optimal distribution of a current density geometrically restricted to specified non-intersecting design surfaces, thereby defining the preferred coil conductor shapes. Instead of boundary integral type methods, which are widely used to design coils, this paper proposes an optimization method for designing multiple layer gradient coils based on a finite element discretization. The topology of the gradient coil is expressed by a scalar stream function. The distribution of the magnetic field inside the computational domain is calculated using the least-squares finite element method. The first-order sensitivity of the objective function is calculated using an adjoint equation method. The numerical operations needed, in order to obtain an effective optimization procedure, are discussed in detail. In order to illustrate the benefit of the proposed optimization method, example gradient coils located on multiple surfaces are computed and characterised.

Keywords Magnetic resonance imaging · Gradient coil design · Stream function · Least-squares finite element method · Adjoint method

1 Introduction

Magnetic resonance imaging (MRI) (Lauterbur 1973) is a commonly used noninvasive technique in radiology used to visualize the structure and function of the body. It can provide detailed images of the interior of the human body over any arbitrary section plane. At the same time, MRI provides better contrast between different soft tissues of the body than CT, making it especially useful in neurological, cardiovascular, and oncological imaging. In an MRI scanner, there are three main parts: a main magnet, three gradient coils, and a radio frequency (RF) transmitter-receiver, called the spectrometer. The main magnet, the largest component of the scanner, is used to generate a strong static and homogeneous magnetic field. The remainder of the scanner is built around it. Gradient coils spatially encode the positions of protons by varying the magnetic field across the imaging volume. The RF transmission system consists of a RF synthesizer, a power amplifier and a transmitting coil. The RF receiver consists of coils, a pre-amplifier and a signal processing system (Marinus and Jacques 2010).

For imaging purposes, gradient coils are used to produce precise non-uniform magnetic fields that vary spatially

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over the imaging volume. In current clinical MRI scanners, three orthogonal gradient magnetic fields, varying linearly with the x, y and z coordinates, are generated by three separate gradient coils, which are denoted G_x, G_y and G_z respectively. The design of a gradient coil can be cast as an optimization problem. A spatial distribution of the magnetic field inside a region of interest (ROI), Ω_{ROI}, is specified, and the goal is to find the optimal distribution of the current density $\mathbf{J}$ on a specified design surface Γ_{coil}. For gradient coils, three orthogonal and separately controllable magnetic field gradients must be generated. The main design goals for gradient coils are the gradient field uniformity, the strength of the field gradient, the switching time, and the thermal performance of the system. Usually, the uniformity of the gradient magnetic field enables an undistorted MRI image; stronger gradients allow for higher resolution of the resulting MRI images; a lower inductance of the gradient coil allows for faster switching and imaging processes; and good thermal performance avoids coil heating through the formation of localised hot-spots. In addition to the above four requirements, active shielding and torque balancing of gradient coils are also considered in most practical designs.

Over the past 30 years, numerous papers have discussed theoretical methodologies for designing gradient coils (Turner 1986, 1988, 1993; Shi and Ludwig 1998; Forbes and Crozier 2004; Forbes et al. 2005; Liu et al. 2007; Peeren 2003; Poole and Bowtell 2007; Marin et al. 2008; Lopez et al. 2009; Hidalgo-Tobon 2010). The established design procedure includes two steps, one to calculate of the spatial distribution of the magnetic field inside an ROI, and the other to search for the optimal distribution of the current density on a pre-specified design surface that represents the coil conductors.

Currently, there are two classes of methods commonly used to implement these optimization procedures: direct optimization methods (DOM) (Turner 1986; Mansfield and Chapman 1986; Turner 1988; Roemer and Hickey 1988; Carlson et al. 1992; Turner 1993; Chronik and Rutt 1998; Forbes and Crozier 2004; Forbes et al. 2005; Lemdiasov and Ludwig 2005; Liu et al. 2007; Poole and Bowtell 2007; Marin et al. 2008) and iterative optimization methods (IOM) (Adamiak et al. 1992; Pissanetzky 1992; Du and Parker 1998; Shi and Ludwig 1998; Peeren 2003; Ungersma et al. 2004; Shvartsman and Steckner 2007; Lopez et al. 2009; Poole et al. 2010; Jia et al. 2011).

The DOM is an optimization algorithm in which the design of a gradient coil can be directly determined through solving a single linear algebraic system. A typical DOM is the target field method (TFM) (Turner 1986, 1988, 1993; Forbes and Crozier 2004; Forbes et al. 2005; Liu et al. 2007), in which a magnetic field is described by the Fourier-Bessel expansion, and by the Fourier transform of the current density on a design surface Γ_{coil}. A gradient coil can be obtained by typically minimizing one of the coil's properties, such as its inductance (Turner 1988), subject to a constraint in which the computed magnetic field is fixed exactly at a finite number of points in a ROI according to a target magnetic field. A linear algebraic system related to this optimization problem can be obtained by introducing Lagrange multipliers for the constraint. Even though the DOM has been quite successful for designing MRI coils, the design methodology still faces challenges. For instance, the current-carrying design surface should be regular (i.e. cylindrical, elliptic cylindrical, or planar). At the same time, the DOMs cannot be extended directly to optimization problems with nonlinear and nonquadratic objective functions (Jia et al. 2011) or nonlinear constraints (Lopez et al. 2009).

For the IOM, the optimal current density distribution can be obtained by solving the optimization problem by starting from a user-chosen initial value (Shi and Ludwig 1998; Peeren 2003; Lopez et al. 2009; Jia et al. 2011). The magnetic field in an IOM is usually calculated by solving a large scale linear algebraic equation system based on the numerically discretized Maxwell equations. In most cases, the DOM has a big advantage from a computational cost point of view. This is one of the reasons why the DOM is has remained the first choice for coil design. However, for the IOM, the computational domain for the magnetic field distribution and the shape of the coil surface Γ_{coil} can both be irregular; moreover, the optimization objective function and design constraints can be chosen to be any physically realizable expression, and hence may also be nonlinear and nonquadratic. The above characteristics provide additional flexibility which can be used in the design of both standard and novel gradient coils.

Boundary integral methods, such as the boundary element method (BEM), the method of moments (MOM), and the fast multipole method (FMM) (Poole and Bowtell 2007; Marin et al. 2008; Lopez et al. 2009; Jia et al. 2011), are commonly used to calculate the magnetostatic problem. The boundary integral methods have the advantage that only material surfaces need to be meshed, rather than the entire computational volume. It is also found that the obtained direct solution of the magnetic field is accurate for both interior and external fields, and that the computational cost is acceptably low especially for the two dimensional case.

Recently, the finite element method (FEM), has gained popularity as a method for the optimization of various engineering problems, including electromagnetics. The FEM can solve both linear and nonlinear electromagnetic problems on regular or piecewise regular domains. Therefore, the FEM holds potential as a method to design novel magnetic coils under more general nonlinear optimization objectives and constraints. However, the computational cost for a three-dimensional finite element calculation of the magnetostatic problem is often more expensive than when using boundary integral methods. For the case that an optimization procedure may take several hundred iterations, the computational cost of the IOM based on the FEM is still considered to be too expensive. In order to use the FEM to design a MRI coil effectively, one of the biggest challenges facing practitioners is, how to increase the optimization efficiency so that designers can obtain an optimized coil in a reasonable time, and this is the issue what we wish to address. Hence, the main goal of this paper is to suggest a method to improve the performance of the IOM based on the FEM, especially focusing on speeding up the coil design procedure.

The remainder of the paper is organized as follows. Section 2 deals with choosing design variables in order to express the surface current density. Section 3 describes a solution of the magnetostatic problem using the least-squares finite element method. Section 4 presents the optimization model and the corresponding sensitivity analysis based on the adjoint method. The detailed numerical skills are discussed to speed up an optimization procedure in Section 5, followed by their numerical implementation in Section 6. Section 7 combines the techniques to address the design of gradient coils whose conductors are restricted to planar or cylindrical multiple-layer surfaces.

2 Design variables for coil optimization

In order to design MRI gradient coils on generalized surfaces, a flexible expression for the surface current density is required. The surface current density **J** is a vector of three components $(J_x, J_y, J_z)^T$ defined on a two-dimensional surface Γ_{coil} embedded in a three-dimensional space. At the same time, the surface current density **J** should also satisfy the divergence-free condition $\nabla_\Gamma \cdot \mathbf{J} = 0$, where $\nabla_\Gamma \cdot$ is the divergence operator restricted to the surface Γ_{coil}. If one chooses three components J_x, J_y and J_z of the surface current density as design variables in a coil optimization model directly, then the divergence-free condition cannot be satisfied automatically when using interpolation functions of a Lagrange finite element to approximate the three surface current density components. Therefore, one has to impose the divergence-free condition as an additional equality constraint on the coil optimization model. As a result, the numerical solution of the magnetostatic problem in the optimization model may not be accurate enough if the equality constraint cannot be satisfied exactly. This situation makes it difficult to directly use the vector **J** as the design variable.

It may be possible to specify a scalar function related to the surface current density **J** as a design variable in order to avoid using this constraint in the optimization model. If the current-carrying surface Γ_{coil} is simply connected, or if every closed curve on Γ_{coil} can be shrunk to a point in a continuous way, there is a scalar function ψ such that the surface current density can be directly expressed as

$$\mathbf{J} = \nabla_\Gamma \times (\psi \boldsymbol{n}) \tag{1}$$

where **n** is the normal vector of the surface Γ_{coil} and $\nabla_\Gamma \times$ is the curl operator on the sheet Γ_{coil}. This is known as the stream function (Pissanetzky 1992; Peeren 2003; Gross and Kotiuga 2004). In this paper, the scalar stream function ψ is chosen as the design variable in the coil optimization model. The surface current density **J** can be calculated by (1) and the divergence condition of the vector **J** is automatically satisfied when the scalar stream function on the surface boundary is specified as a constant (Peeren 2003). In the numerical calculation of the optimization model, the scalar stream function ψ is interpolated using a linear Lagrangian (nodal) finite element.

A real surface coil should be a discrete winding with a constant conductor current, instead of a sheet conductor with a continuous current density distribution. Hence, an additional step is necessary in order to transform the expression of the continuous surface current to a discrete winding. For a surface current density **J**, there is a equivalent expression for the divergence free condition (Peeren 2003)

$$C_i = \int_C \mathbf{J} \cdot (dl \times \mathbf{n}) = \int_C (\mathbf{n} \times \mathbf{J})\, dl = 0 \tag{2}$$

for any closed contour C_i on the stream function surface. This property provides a way to express the discrete coils given a surface with a continuous stream function. In principle, there are four steps which are needed to generate the individual wire windings for this case (Fig. 1):

1. Choose a suitable number N so that a constant electric current flow through the discrete coil can be decided as $(\psi_{\max} - \psi_{\min})/N$.
2. Use the contour lines of the stream function to generate separate loops.
3. Assign the width H of the conductor and extend $H/2$ distance along the normal direction to both sides of contour lines. The conductor width constraint H should be reasonable, so that any two individual conductors do not intersect each other.
4. Cuts are introduced into each one of the individual concentric current loops, and interloop connections are introduced among each of the adjacent current loops at these cuts, so as to effectively form a single continuous current conducting path with a spiral topology.

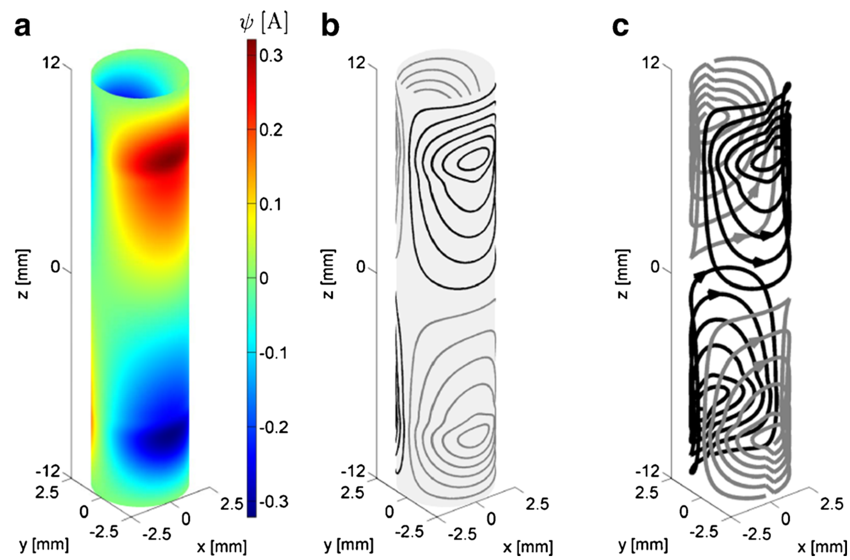

Fig. 1 Coil discretization procedure leading from a stream function surface to an individual wire winding. (**a**) Stream surface expression (**b**) contour line expression (**c**) individual wire winding

For the later examples presented in this paper, the contour lines from the second step are used to illustrate the layout of the gradient coils.

3 Solving magnetostatic problem using least-squares FEM

The differential form of the original Maxwell equations has only first-order differential operators (Jackson 1998). When designing a gradient coil using an IOM, we mainly consider the steady-state magnetic phenomena produced by the gradient coil. This situation means that we only need to solve the magnetostatic equations simplified from the Maxwell equations (Jackson 1998; Jin 2002). These static equations are first-order partial differential equations (PDEs). The magnetic field **B** is the main unknown in these static equations for a specified computational domain with suitable boundary conditions. In this paper, a numerical discretization of the magnetostatic problem is implemented using the least-squares finite element method (LSFEM), instead of the standard Galerkin method via a magnetic vector potential.

As illustrated by Ern and Guermond (Ern and Guermond 2004), the numerical solution of a standard Galerkin method for a first-order PDE can contain spurious oscillations and hence cannot accurately approximate the solution of the first-order PDE. In order to solve the static equations using a standard Galerkin discretization, a magnetic vector potential **A**, which is defined as $\mu_0^{-1}\mathbf{B} = \nabla \times \mathbf{A}$ and $\nabla \cdot \mathbf{A} = 0$, is often used to transform the first-order equations into a second-order elliptical equation (Jackson 1998; Jin 2002). Then the second-order elliptical equation can be solved accurately using the standard Galerkin method. However, when optimizing the layout of a gradient coil, the spatial distribution of the magnetic field **B**, instead of the magnetic vector potential **A**, is used in typical objective functions of the optimization model. Therefore, it is natural to inquire whether **B** can be solved directly using an alternative numerical discretization method. The LSFEM is a method to minimize a least-squares functional with first order differential operators (Jiang 1998; Bergstrom 2002; Bochev and Gunzburger 2009). For a first-order PDE, the LSFEM solves a symmetric and positive definite (SPD) algebraic problem. Because there is no magnetic material discontinuity for gradient coil design, the LSFEM can compute the magnetic field **B** directly using Lagrange-type finite elements. At the same time, the discretized SPD global stiffness matrix is well suited for fast sensitivity analysis, which is discussed in Section 5.

3.1 Magnetostatics

For the physical problem considered in this paper, only direct electrical current are necessary, and these can be assumed as surface currents instead of a volume currents, when solving the distribution of magnetic field generated by a MRI gradient coil. Based on this assumption, the Maxwell equations can be simplified, since there is no electrostatic charge and no electric field, and the magnetic field is constant with respect to time. As a result, we obtain the following magnetostatic equations:

$$\begin{aligned} \nabla \times {\mu_0}^{-1}\mathbf{B} &= 0 \quad \text{in } \Omega_1 \text{ and } \Omega_2 \\ \nabla \cdot \mathbf{B} &= 0 \quad \text{in } \Omega_1 \text{ and } \Omega_2 \end{aligned} \tag{3}$$

where **B** is the magnetic field, μ_0 is the permeability of free space, Ω_2 is a subdomain which includes the ROI, and Ω_1 is a subdomain which is located outside of the subdomain Ω_2. The union of the two subdomains Ω_1 and Ω_2

constitutes the computational domain (Fig. 2). The corresponding boundary and interface conditions associated with the computational domain are

$$\mathbf{B}\cdot\mathbf{n} = 0 \text{ on } \Gamma$$
$$\left[\mu_0^{-1}\mathbf{B}\times\mathbf{n}\right] = \mathbf{J},\ [\mathbf{B}\cdot\mathbf{n}] = 0 \text{ on } \Gamma_{coil} \tag{4}$$

where Γ is the outer boundary of the subdomain Ω_1, Γ_{coil} is the interface between the subdomains Ω_1 and Ω_2, $\mathbf{n}$ is the exterior unit normal vector on the boundary Γ or the unit vector normal to the interface Γ_{coil}, $\mathbf{J}$ is a surface current density, and $[u(x)] := \lim_{s\to 0^+} u(x+s\mathbf{n}) - u(x-s\mathbf{n})$ with $x \in \Gamma_{\text{coil}}$ denotes the jump of a function u across the interface Γ_{coil}.

3.2 The least-squares finite element method

The least-squares finite element method (Bergström 2002; Bochev and Gunzburger 2009) is employed to solve the above magnetostatic problem with suitable boundary conditions. A least-squares functional in the LSFEM is defined as the sum of residual norm squares of (3) and (4):

$$\begin{aligned} I(B) = &\sum_{i=1}^{2}\left(\left\|\mu_0(\nabla\times(\mu_0^{-1}\mathbf{B}))\right\|_{\Omega_i}^2 + \left\|\nabla\cdot\mathbf{B}\right\|_{\Omega_i}^2\right) \\ &+ \left\|h^{-1/2}\mu_0\left[\mu_0^{-1}\mathbf{B}\times\mathbf{n}-\mathbf{J}\right]\right\|_{\Gamma_{\text{coil}}}^2 \\ &+ \left\|h^{-1/2}[\mathbf{B}\cdot\mathbf{n}]\right\|_{\Gamma_{\text{coil}}}^2 + \left\|h^{-1/2}[\mathbf{B}\cdot\mathbf{n}]\right\|_{\Gamma}^2 \end{aligned} \tag{5}$$

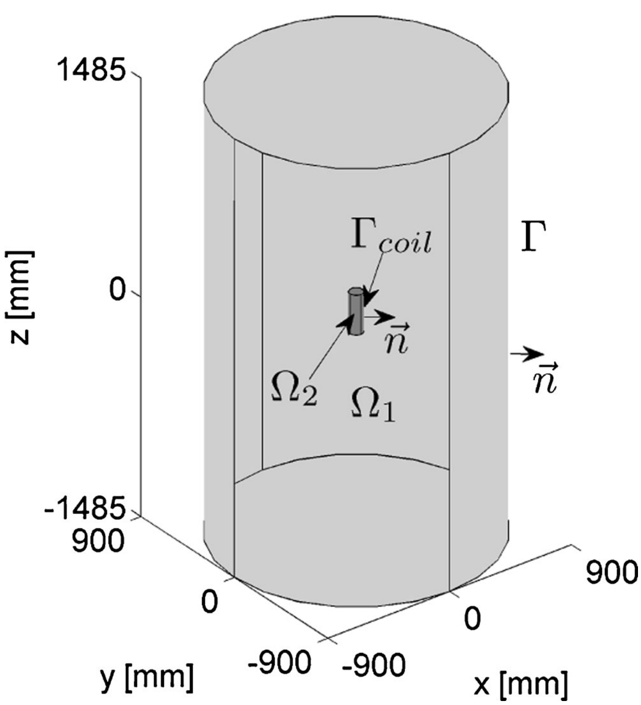

Fig. 2 The computational domain and coil position for the LSFEM

where h denotes the size of a finite element, $\|u\|$ is the L^2 norm of the function u. A necessary condition for the above minimal problem is $\lim_{t\to 0}\frac{\partial}{\partial t}I(\mathbf{B}+t\widetilde{\mathbf{B}}) = 0$, where $\widetilde{\mathbf{B}}$ is a test function. This condition results in the following variational problem: find a $\mathbf{B}\in V$ such that $a(\mathbf{B},\widetilde{\mathbf{B}}) = l(\widetilde{\mathbf{B}})$ for all $\widetilde{\mathbf{B}}\in V = H^1(\Omega_1)\oplus H^1(\Omega_2)$, where

$$\begin{aligned} a(\mathbf{B},\widetilde{\mathbf{B}}) &= \sum_{i=1}^{2}((\nabla\times\mathbf{B},\nabla\times\widetilde{\mathbf{B}})_{\Omega_i} + (\nabla\cdot\mathbf{B},\nabla\cdot\widetilde{\mathbf{B}})_{\Omega_i}) \\ &\quad + h^{-1}(([\mathbf{B}\times\mathbf{n}],\widetilde{\mathbf{B}}\times\mathbf{n})_{\Gamma_{\text{coil}}} + ([\mathbf{B}\cdot\mathbf{n}],\widetilde{\mathbf{B}}\cdot\mathbf{n})_{\Gamma}) \\ l(\widetilde{\mathbf{B}}) &= h^{-1}(\mathbf{J},\widetilde{\mathbf{B}}\times\mathbf{n})_{\Gamma_{\text{coil}}} \end{aligned} \tag{6}$$

and (A, B) denotes the inner product of $L^2(\Omega_1)$ or $L^2(\Omega_2)$. Let V_h be a standard linear Lagrange finite element space of V, the goal of the LSFEM is to obtain a numerical solution $\mathbf{B}_h \in V_h$ such that $a(\mathbf{B}_h,\widetilde{\mathbf{B}}_h) = l(\widetilde{\mathbf{B}}_h)$ for all $\widetilde{\mathbf{B}}_h \in V_h$.

3.3 Comparing the accuracy of numerical solutions for the magnetic field

For gradient coil design, it is necessary to calculate the value and the spatial derivative of B. For the standard Galerkin discretization, the magnetic vector potential $\mathbf{A}$ is the unknown of a second-order elliptical equation derived from the magnetostatic equation, and the magnetic field B is calculated as the first-order numerical derivative of the magnetic potential $\mathbf{A}$ (Jin 2002). This means that the accuracy of the magnetic field B is one order less than the accuracy of the magnetic potential $\mathbf{A}$. Therefore, it is unusual to directly calculate the spatial derivative of B if a linear finite element is used to discretize the distribution of the magnetic potential $\mathbf{A}$. In this case, a second-order finite element needs to be used in order to calculate a second-order numerical derivative from a solution for $\mathbf{A}$. For the LSFEM, the value of the magnetic field $\mathbf{B}$ is used directly as nodal unknown. When the magnetic field $\mathbf{B}$ is discretized using a linear element, the spatial derivative of B is constant within each element.

Figure 3 depicts the solution of magnetic field B_z inside a cylinder for which the Maxwell coil resides on the cylindrical surface using the LSFEM with linear Lagrange element, the standard Galerkin method with linear vector finite element (VFM) and quadratic VFM, and the analytical method, respectively. The computational domain is discretized using the same number of finite elements. All the results are obtained using the Pardiso solver of the commercial FEM package *Comsol* (http://www.comsol.com). Figure 3b shows that the numerical solutions $\mathbf{B}$ have similar accuracy for the standard Galerkin method with quadratic VFM, and the LSFEM with linear Lagrange element. The standard Galerkin method with linear VFM has considerably larger error. Although these errors could be reduced by refining the mesh, this remedy will more approach the

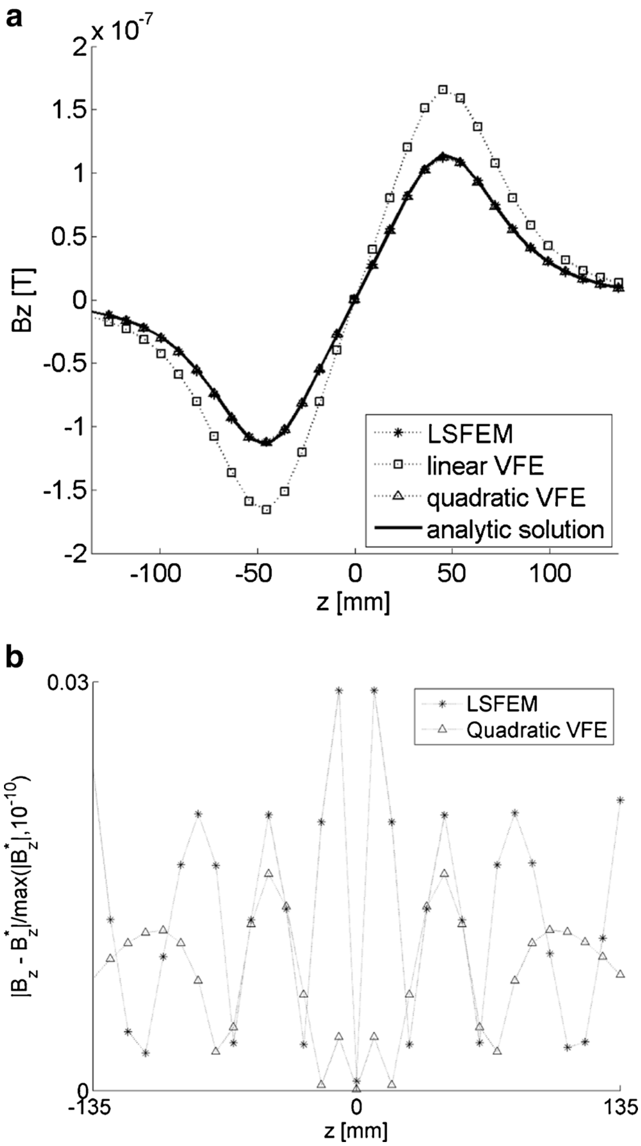

Fig. 3 Comparison of B_z (**a**) and relative error of B_z (**b**) using different numerical and analytic methods for a Maxwell coil pair. Here, B_z^* denotes the analytical solution for the Maxwell coil

bottleneck of computational cost for a large-scale optimization procedures. Table 1 shows the computational time used to calculate the magnetic field distribution. It shows that the computational cost for the LSFEM with linear Lagrange element is much lower than for the standard Galerkin method with quadratic VFM. Therefore, the LSFEM with linear Lagrange element is found to be the most efficient choice in term of balance between computational cost and numerical accuracy.

4 Optimization model and sensitivity analysis

Gradient coils which are located on a surface Γ_{coil} are designed to generate a specified magnetic field within the

Table 1 The CPU time for solving the magnetostatic problem

	DOFs (thousands)	CPU time (seconds)
LSFEM	148	146.8
linear VFM	184	152
quadratic VFM	1456	9816

region of interest Ω_{ROI}. For typical MRI gradient coils design, the objectives may include not only the target magnetic field, but also other physical quantities which need to be minimized or maximized. The most used quantities include the inductance, magnetic energy and the mechanical torque. For the case that there are multiple optimization objectives, there are a couple of choices to establish an optimization model. A straightforward choice is the weighted sum objective method, where all the optimization objectives are summed together with different weights. One could obtain a list of optimal solution which is called Pareto optimal by changing the value of weights. Alternatively, one design objective can be chosen as the main objective and the other design necessaries are included into the Lagrangian form of optimization model by using the augmented Lagrangian method. Usually, designers do not know the available minimum value of the coil inductance or magnetic energy, the objective function of a target magnetic field is transformed as an equality constraint using the Lagrange multiplier method. Thirdly, the constrained optimization model can be used, where the most important objectives are chosen as the optimization objective, and the other objectives are transformed into constrained conditions with user specified lower and upper bounds. Because the main purpose of this paper focuses on the simplification of gradient coils design based on the least squares finite element method, a single-objective optimization model is presented and the corresponding sensitivity analysis based on the adjoint method is derived.

4.1 Optimization model

In this paper, we consider the design of G_x gradient coils which can generate a linearly varying z-component B_z of a magnetic field along the x-direction. There are two choices to express the distribution of target gradient field B_z^*. The first is $B_z^* = kx + b$ where k and b are constants, and the second is $\partial B_z^*/\partial x = k$. From an analytical analysis point of view, there is no difference between these two choices when B_z is smooth enough. However, a numerical solution of the magnetic field B_z will differ from the analytical case when the FEM is used. As shown in the Section 3.3, the LSFEM numerical solutions B_z is smooth and accurate enough when compared with the analytical solution. Therefore, we choose the optimization model as

$$\begin{aligned} \text{Min:} \quad & F = \int_{\Omega_{\text{ROI}}} \left(\frac{\partial B_z}{\partial x} - k \right)^2 d\Omega_{ROI} \\ \text{s.t.} \quad & a(\mathbf{B}, \widetilde{\mathbf{B}}) = l(\widetilde{\mathbf{B}}) \end{aligned} \tag{7}$$

where the constraint equation is the least-squares finite element discretization of (6). The design variable of the optimization model is the stream function ψ, and the surface current density $\mathbf{J}$ in (6) is expressed in terms of ψ using (1).

4.2 Sensitivity analysis using the adjoint method

Sensitivity analysis for the optimization involves the calculation of changes in the objective function which result from changes in the design variable. In general, several factors must be considered in order to choose a suitable sensitivity analysis method: the accuracy of the calculation, the computational effort involved, and the complexity of the implementation. For the numerical optimization of gradient coils in which the numerical solution of the magnetic field is implicitly dependent on the design variable of the optimization model, the adjoint equation method is used in order to efficiently calculate the first-order sensitivity vector of the objective function.

For the optimization model in (7), the corresponding discretized Lagrangian model using the LSFEM can be expressed as

$$L = F\big(\mathbf{B}(\psi), \psi\big) + \lambda^T(\mathbf{J} - \mathbf{KB}), \tag{8}$$

where $\mathbf{B}$ is the unknown vector of the magnetic field, ψ is the stream function, λ is a Lagrange multiplier vector, $\mathbf{J}$ is the vector of surface current density, and $\mathbf{K}$ is the discretized global stiffness matrix from the LSFEM. The first order sensitivity, based on the chain rule for differentiation, is

$$\frac{\partial L}{\partial \psi} = \frac{\partial F}{\partial \psi} + \frac{\partial F}{\partial \mathbf{B}}\frac{\partial \mathbf{B}}{\partial \psi} + \lambda^T\frac{\partial \mathbf{J}}{\partial \psi} - \lambda^T\frac{\partial \mathbf{K}}{\partial \psi}\mathbf{B} - \lambda^T\mathbf{K}\frac{\partial \mathbf{B}}{\partial \psi}. \tag{9}$$

Because the magnetostatic equation $\mathbf{KB} = \mathbf{J}$ is solved before sensitivity analysis is performed, the Lagrange multiplier λ is an arbitrary, fixed vector. Hence we can rearrange the sensitivity (9) as

$$\frac{\partial L}{\partial \psi} = \frac{\partial F}{\partial \psi} + \lambda^T\frac{\partial \mathbf{J}}{\partial \psi} - \lambda^T\frac{\partial \mathbf{K}}{\partial \psi}\mathbf{B} \tag{10}$$

where the terms which include $\partial\mathbf{B}/\partial\psi$ are removed by solving for λ from the adjoint equation

$$\mathbf{K}^T\lambda = \left(\frac{\partial F}{\partial \mathbf{B}}\right)^T \tag{11}$$

Here, $\mathbf{K}^T$ denotes the transpose of $\mathbf{K}$. This completes the derivation of the first-order sensitivity of the optimization objective function. The detailed numerical implementation of the above system in the commercial FEM software *Comsol* was shown by Olesen et al. in Olesen et al. (2006).

5 Speedup optimization procedure using matrix factorization

When the FEM is used to implement the design of an MRI gradient coil, the computational cost is extremely high for solving the magnetostatic equation and sensitivity vector, because a three-dimensional finite element analysis may have a discretized matrix with several hundred thousand DOFs. Considering that an optimization may take several hundreds of iterations, a whole optimization procedure for a three dimensional coil design may take more than one day or even longer when using the latest powerful workstation. In this section, a speedup technique for the optimization procedure is proposed based on surface design variables.

The optimization of a gradient coil includes several key steps which are run sequentially. The main computational cost for an iteration is due to the finite element and sensitivity analysis steps. When the LSFEM is used to discretize a design domain, the discretized stiffness matrix $\mathbf{K}$ does not alter during the iterations, and merely needs to be assembled once for all iterative loops. Therefore, only the vector on the right-hand side of the discretized equation needs to be updated in each iteration. Based on this fact, a fast solving strategy is used to speed up the optimization procedure.

We note that the discretized global stiffness matrix $\mathbf{K}$ is symmetrical and positive definite for the case when the LSFEM with the Lagrange element is used to solve the magnetostatic equation (7). Hence the global stiffness matrix is assembled and immediately decomposed once using a suitable linear algebra solver. For example, when solving the linear algebra equations $\mathbf{KB}_i = \mathbf{J}_i$ using a direct solver during iteration i, one can use the predetermined Cholesky factorization $\mathbf{K} = \mathbf{L^TDL}$, where $\mathbf{L}$ is a lower triangular matrix and $\mathbf{D}$ is a diagonal matrix. After the decomposition step and during each iteration i, the vector of $\mathbf{B}_i$ is solved efficiently by using a back-substitution step. Table 2 presents the computational time required for the decomposition and substitution steps, which shows that, relatively speaking, the latter step requires much less time. In addition, the saved matrix can also be reused for a variety

Table 2 The CPU time and memory usage for matrix decomposition and substitute

Dofs (thousands)	283.5	148.524	84.425
Decomposition time (seconds)	319.1	117.1	39.4
Substitution time (seconds)	5.1	1.99	0.86
Memory usage (gigabytes)	7	3	1.3

optimization objective function calculations when the computational domain and its corresponding mesh discretization are kept unchanged.

For the case of a computer with and AMD64 CPU (2-core 2.6G) processor, one optimization iteration merely takes 3 seconds when the dimension of the global stiffness matrix is 84425, and less than a minute when the dimension of the global stiffness matrix is about 283500. Based on the numerical experiments, it was found that discretizations leading to global stiffness matrix dimensions of 250000 are usually sufficient for typical gradient coil design tasks. A potential limitation for this strategy might be the computer memory used to save the decomposed matrices (Table 2). Certainly, an iterative solver, such as the symmetric successive over-relaxation preconditioned conjugate gradient method (SSOR-CGM), requires much less memory than needed for Cholesky factorization. However, the computational time of substitution is still 10 times faster than that of SSOR-CGM (Table 3). Therefore, the direct solver with the routines CHOLMOD (Davis and Hager 2005; Chen et al. 2008), whereby the METIS (Karypis and Kumar 1998) is used to re-order the sparse matrix, is employed to perform the Cholesky factorization and substitution in this paper.

6 Numerical implementation of coil design

In order to design an MRI gradient coil using the method proposed above, three implementation issues need to be addressed. These are now discussed.

6.1 Maintaining the symmetry of coils

For a given optimization objective function, the optimal layout of MRI coils may contain symmetries. When using the DOM, the active design surface can be chosen as a subset of Γ_{coil} based on a symmetry property. However, the choice of symmetry may differ among coils based on different specified magnetic fields and design domains. Instead of choosing part of Γ_{coil} as the design surface, the whole cylinder surface Γ_{coil} is used for all numerical examples presented in this paper. Theoretically, the symmetry of an optimized coil should be enforced automatically during the optimization procedure. However, the layout of the coil may deviate from the perfect symmetrical case because of numerical errors of the magnetic field **B** and the sensitivity vector. In this paper, symmetry is maintained by adding additional constraints for the original sensitivity vector in (10). Even though this method seems to be computationally more expensive than when only part of Γ_{coil} is chosen as the design surface, the benefit of not doing so is that the optimization code can be extended to more general coil designs by changing only the expression of the optimization objective and the symmetry constraint for the original sensitivity vector. To illustrate, Fig. 4 shows examples of a cylindrical linear G_x-gradient coil with and without the imposition of symmetrical properties.

Table 3 The CPU time for iterative solver named SSOR-CGM

Dofs (thousands)	283.5	148.524	84.425
Computational time (seconds)	78.5	30.2	15.5

6.2 Choosing the descent direction

The sensitivity vector is obtained using the sensitivity analysis method described in Section 4.2. The steepest descent (SD) method and the conjugate gradient (CG) method are the most commonly used first order methods to update the design variables. Alternatively, one can also use the quasi-Newton method, for example, the limited memory BFGS (L-BFGS) method. In this paper, all three methods

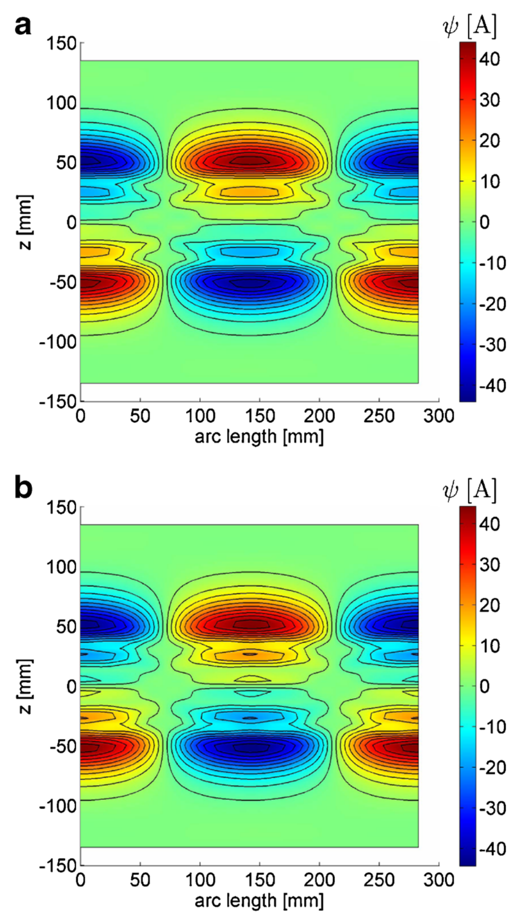

Fig. 4 Stream function and its contours on the developed current-carrying surface with enforced symmetry (**a**) and without enforced symmetry (**b**) for a G_x cylindrical gradient coil

with linear search algorithms satisfying the Wolfe conditions (Nocedal and Wright 1999) are used to perform the gradient coil optimizations. In order to obtain a reasonable optimal layout, two stopping criteria are used. The first is a set of three conditions taken from Gill et al. (Gill et al. 1982) which are expressed in the following formula:

$$|F_k - F_{k-1}| \quad < \epsilon(1 + |F_k|); \tag{12}$$

$$\|\psi_k - \psi_{k-1}\| < \sqrt{\epsilon}(1 + \|\psi_k\|); \tag{13}$$

$$\|g_k\| \quad \leq \sqrt[3]{\epsilon}(1 + |F_k|). \tag{14}$$

Here, F_k, ψ_k and g_k are values of the objective function F, the stream function ψ, and the $\partial L/\partial\psi$ at the kth optimization step, respectively. Users can control the desired accuracy by specifying the tolerance parameter ϵ, which is preset to 10^{-9} for all examples of this paper. Conditions (12) and (13) are employed to check whether the objective function and design variable converge. At the same time, the necessary first-order optimization condition is considered for the third condition (14). The second stopping criterion is the total number of objective function evaluations (TNOFE), which is set to 500. Figure 5 shows the convergence for the design of a single layer cylinder gradient coil, which can generate a linearly varying magnetic field along the x axis inside the ROI. This figure shows that the L-BFGS method performs much better than the SD and CG methods, both in terms of the total number of iteration loops and the total number of objective function evaluations.

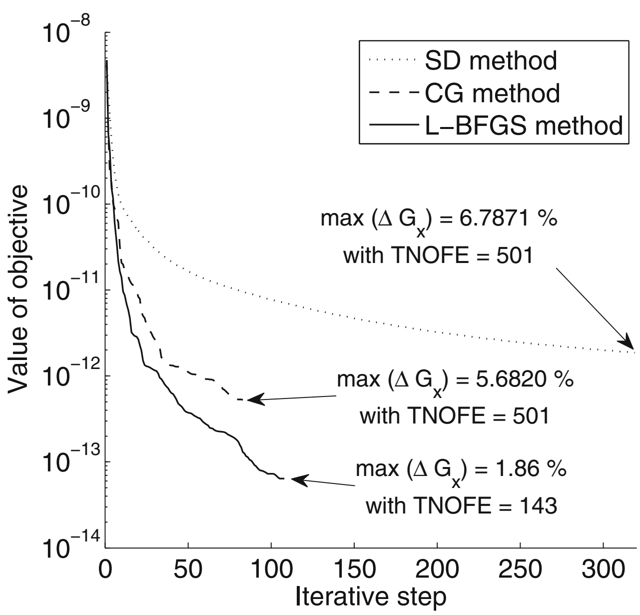

Fig. 5 Comparison of the iterative convergence for three optimization methods

To satisfy requirements for MRI engineering, the linear gradient deviation of the magnetic field $\triangle G_x$, which is defined as

$$\triangle G_x = \left|\frac{\partial B_z/\partial x - \partial B_z^*/\partial x}{\partial B_z^*/\partial x}\right|. \tag{15}$$

also needs to be checked to evaluate the overall performance of the designed gradient coils. For the later numerical examples presented in this paper, the LBFGS method is used to update the design variables, and (12)–(14) together with a $\triangle G_x$-value of less than 5 % are chosen as stopping criteria.

6.3 Smoothing sensitivities

The MRI coil optimization problem is usually ill posed. There are several regularization methods, for example the Tikhonov regularization technique, that are suitable to overcome the aforementioned problem. For an ill-posed problem in structural topology optimization, a sensitivity filter technique has been widely used to prevent numerical instabilities (Sigmund and Petersson 1998). The filter of sensitivities is implemented by modifying the sensitivity at one node with a weighted average of sensitivities in the neighbourhood of this point. In this paper, a filter is chosen as

$$\frac{\partial L}{\partial \psi_k} = \frac{1}{\sum_{i=1}^{N} H_i} \sum_{i=1}^{N} H_i \frac{\partial L}{\partial \psi_i}, k = 1, \ldots, N, \tag{16}$$

where N is the total number of nodes in the mesh and weights H_i are

$$H_i = \begin{cases} r - \mathrm{dist}(k,i), & \text{if } \mathrm{dist}(k,i) \leq r; \\ 0, & \text{otherwise.} \end{cases}$$

Here, r is the filter radius and $\mathrm{dist}(k,i)$ is the distance between the kth and ith nodes. The larger the filter radius is, the smoother the resulting stream function will be. However, over-smoothing of the stream function may result in a large residual of the objective function and worse performance of the optimized gradient coil. Generally, the filter radius is problem-dependent and can be determined by numerical experiments. In this paper, the filter radius is 3.5 times and 3.0 times the shortest length of element edges for an inner and outer layer coil in order to achieve the best compromise and obtain a relatively simple topology of the resulting optimized coils.

7 Design G_x gradient coils on cylindrical and planar surfaces

The G_x cylindrical gradient coils, which generate constant gradients of magnetic field along the x axis inside the ROI, are used to illustrate the effect of the design method presented here. When calculating the magnetic field generated by the G_x gradient coil using finite element method, we must consider how to best choose the computational domain and its boundary conditions, because the calculation of the magnetic field computational problem actually has an open boundary, while the LSFEM can only be computed over a bounded domain. Hence, a limited computational domain is used to approximate the magnetic field distribution inside an unbounded domain. In the examples, the computational domain is limited to a cylinder with 900 mm radius and 2970 mm height.

As shown in Fig. 2, the current-carrying surface Γ_{coil} is located between Ω_1 and Ω_2, where Ω_2 is a cylinder with 45 mm radius and 270 mm height. The ROI, which is not shown Fig. 2, is a cylinder with 30mm radius and 42 mm height located inside of Ω_2. The computational domain is discretized using the linear hexahedron element with 89,610 nodes, and the boundary condition on the outside boundary Γ of Ω_1 is set according to (4). This approach can be employed to handle an open boundary problem with truncation errors. Figure 3b demonstrates that this extra error remains less than 5 percent when compared to an analytic solution for a Maxwell coil pair, which is acceptable for an engineering application. The target gradient strength k in the center of ROI is 10 mT/m. Figure 6 shows the stream function and the optimal layout of the G_x coil on a single cylindrical surface Γ_{coil}. The maximum deviation $\triangle G_x$ is 1.86 % corresponding to a maximum absolute value of the stream function ψ of 40. Figure 7 shows the distribution of magnetic field B_z in the ROI.

Multiple-layer gradient coils may achieve a larger gradient under a given maximum input current, or the same magnetic gradient under a lower input current, than a single-layer coil (Bowtell and Robyr 1998; Leggett et al. 2003). The numerical optimization method presented in Section 4 can be extended from the design of a single-layer coil to multiple-layer coils straightforwardly. When compared with the single-layer coil design, the only difference for a multiple-layer coil design is that the design variables are located on additional design surfaces. When a computational domain is meshed so that the design surfaces coincide with the mesh edges, the design variables can be specified on these surfaces directly. Compared to the single-layer coil design method, the computational cost of optimization is similar for multiple layer coil design when the number of

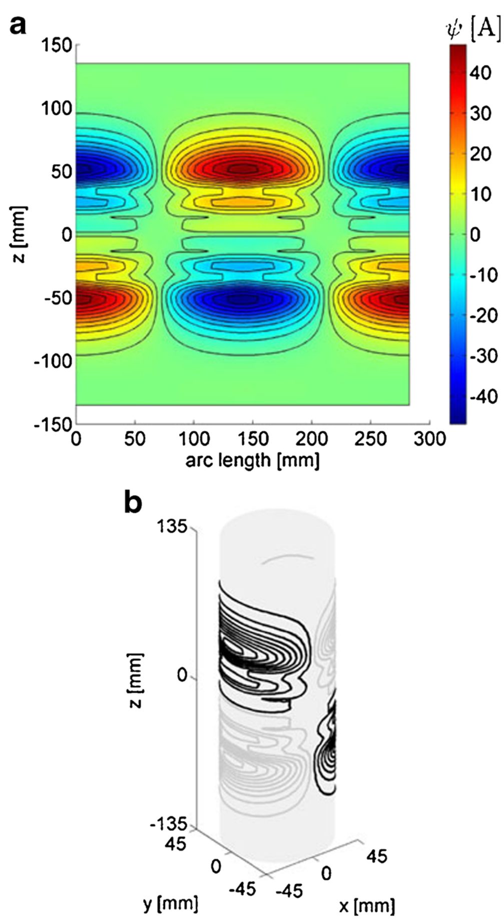

Fig. 6 Stream function contours (**a**) on the developed current-carrying surface and its corresponding coil layout (**b**) for a G_x cylindrical gradient coil using the L-BFGS method

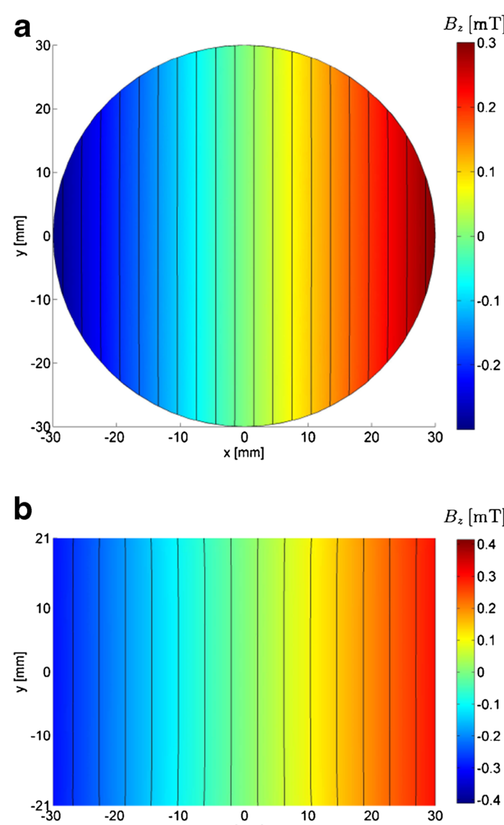

Fig. 7 Magnetic field B_z contours on the $z = 0$ plane (**a**) and the $y = 0$ plane (**b**) for the ROI

DOFs used to discretize the computational domain is similar. This represents an advantage w.r.t. the DOM, in which the computational complexity increases with the number of design variables and the number of coil layers.

As for the single layer coil design example, the target gradient strength in the center of the ROI is required to be 10 mT/m. Figure 8 shows a design of cylinder G_x gradient coils defined on two concentric cylinder surfaces with different radii. Here, the maximum of $\triangle G_x$ is 3.58 % corresponding to the maximum absolute value of the stream function ψ of 20. Because the LSFEM can deal with any shaped computational domain using the piecewise discretization methodology, the design surfaces for multiple-layer coils need not have the same shape. Figure 9 shows a design of the G_x gradient coil on a cylinder and a planar surface. Here, the maximum of $\triangle G_x$ is 4.74 % corresponding to a maximum absolute value of the stream function ψ of 20.

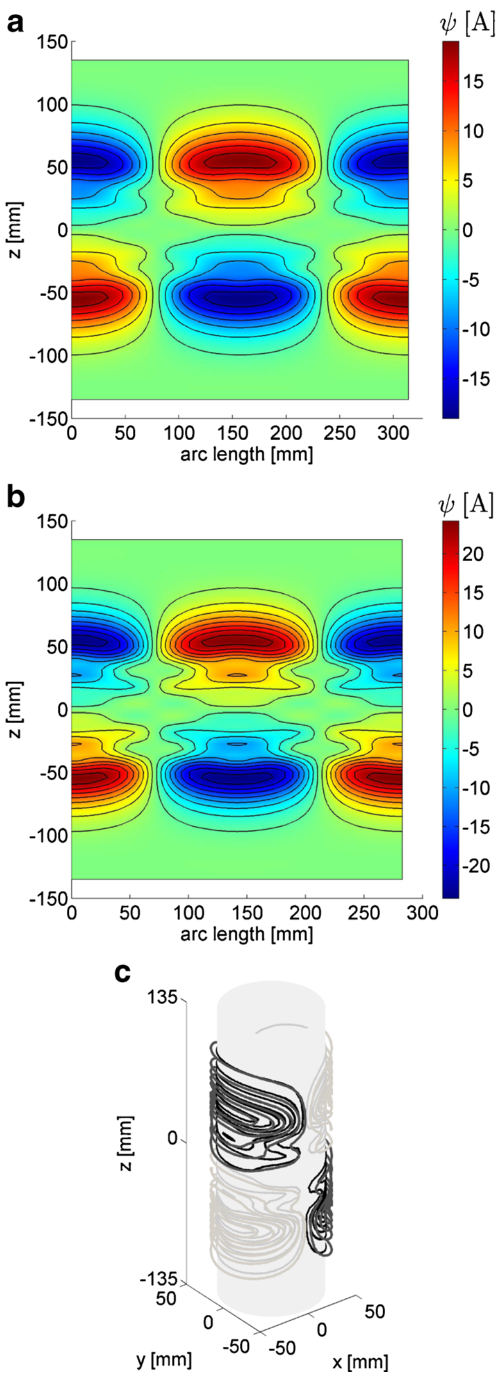

Fig. 8 Design of the G_x gradient coil using two cylindrical surface coils. Stream function contours on the developed outer (**a**) and inner (**b**) current-carrying surfaces are shown. **c** Design layout of the G_x gradient coil on the two concentric cylinder surfaces

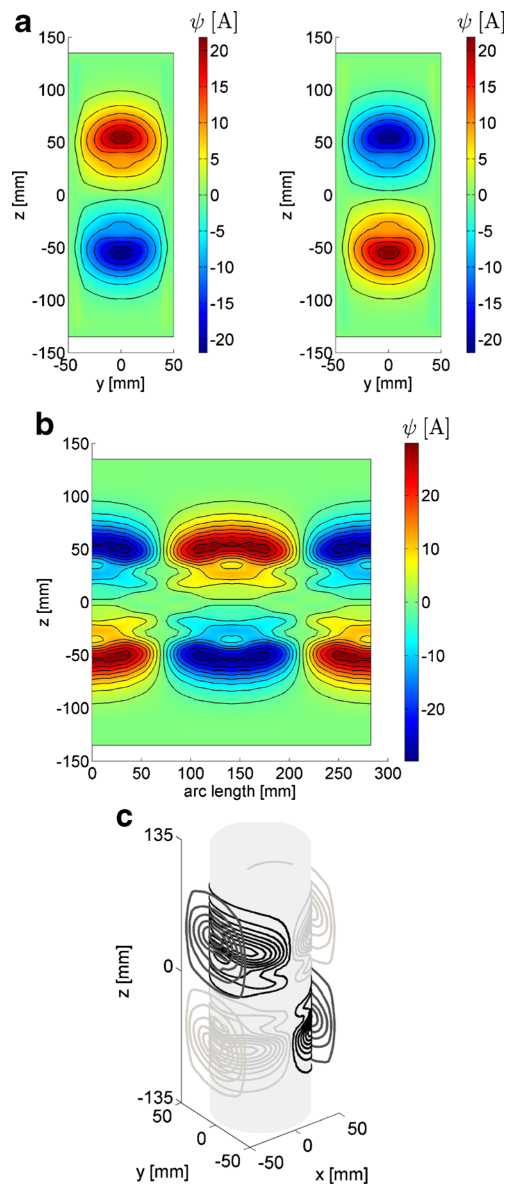

Fig. 9 Design of the G_x gradient coil using one cylindrical surface coil and one planar coil. Stream function contours on the outer current-carrying plane (**a**) and on the developed inner current-carrying surface (**b**) are shown. **c** Design layout of the G_x gradient coil on the planar and cylinder surfaces

In these two multiple-layer coils, the maximum absolute value of the stream function is about half the value of the steam function for the single layer coil, which means that the electric current flowing through each wire is approximately half of the value shown in Fig. 6. The possibility to add coil layers provides an additional degree of freedom for designers to balance the coil performance between the magnetic field distribution and the resulting temperature field distribution.

8 Conclusion

The finite element method is a versatile numerical discretization technique for engineering simulation and design, as demonstrated by an efficient implementation of MRI gradient coil design using the LSFEM. When compared to the standard Galerkin method, discretization of the magnetostatic equation using the LSFEM leads not only to an accurate numerical solution of the magnetic field, but also to a positive definite global stiffness matrix, which can be factorized once and reused repeatedly for a number of tasks during optimization. Therefore, the computational time required for the whole optimization procedure speeds up to 100 times faster than methods based on the standard Galerkin finite element method. Even though the computational cost of gradient coil design using the LSFEM is still more expensive than boundary integral type methods, this work significantly narrows the gap among them.

The presented multiple-layer coil examples show the potential to use the finite element method as a unified numerical solver to design a variety of MRI gradient coils, such as the simultaneous design of gradient coils along the three orthogonal directions, and the simultaneous design of gradient and shielding coils.

Since the finite element method can solve both linear and nonlinear physical problems, it provides the ability to perform multiple-objective design tasks, such as optimization of coil's inductance, coil's deformation, its eigenmode shapes (or eigen-frequency) due to electromagnetic forces, the eddy currents, and the resultant heating of coils.

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