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New delay-dependent stability for systems with interval time-varying delay

***Abstract**. This paper is concerned with the stability of systems with interval time-varying delay. By employing a new and tighter integral inequality and constructing an appropriate type of Lyapunov functional, the improved delay-dependent stability criteria are derived. Because neither any model transformation nor free weighting matrices are employed in the theoretical derivation, the developed stability criteria simplify and improve the existing stability conditions. Numerical examples are given to demonstrate the effectiveness and the benefits of the proposed methods.*

***Streszczenie**. Analizowano stabilność systemów z interwałowym zmiennym w czasie opóźnieniem. Wyprowadzono nowe kryteria stabilności bazując na funkcjonale Lapunowa. (**Nowy model stabilności dla systemów z interwałowym zmiennym w czasie opóźnieniem**)*

Keywords: stability; interval time-varying delay; delay-dependent.
Słowa kluczowe: stabilność, kryteria Lapunowa, czasowo zależne opóźnienie.

Introduction

Time delays are frequently encountered in real control systems. The existence of time delays can often lead to instability and performance deterioration if not dealt with properly. The analysis of the stability of dynamic control systems with delays and the synthesis of controllers for them are important both in theory and practice, and are thus of interest to a great number of researchers [1, 2].

Stability criteria for delay systems can be classified into two categories: delay-independent and delay-dependent criteria[3-19]. Generally speaking, the delay-dependent stability conditions are less conservative than delay-independent ones especially when the delay is small. The development of the technologies for delay-dependent stability analysis has been focusing on effective reduction of the conservation of the stability condition. As far as delay-dependent stability is concerned, there are roughly two approaches, namely the frequency-domain one and the time-domain one. As to the time-domain approach, Lyapunov functional is a powerful tool, which can deal with time-varying delays.

Recently, with the rapid advancement of teleoperation [10] and networked control systems [11], robust stability of uncertain systems with interval time-varying delay has attracted extensive research interest. The stability of such kinds of systems was first investigated in reference [12] using a discretized Lyapunov-Krasovskii approach, where the derivative bound of the interval time-varying delay is needed. Afterwards, there are also some achievements obtained in this field. Many existing methods have employed the model transformation technique or Moon's inequality for bounding cross terms. Such treatment of cross terms is a potential source of conservativeness [6]. Therefore, much effort has been devoted more recently to the development of the free weighting matrices method, e.g., [4], [5], [7]–[9], [13], [15]. The free weighting matrices method has been shown to be less conservative than previous methods [8], [9], such as the model transformation technique and Park's inequality methods applied in [7]. However, it has been noticed that using too many free weighting matrix variables will complicate system synthesis and, consequently, lead to significant computational demand [15].Discarding both the model transformation technique and the free weighting method, this paper will develop a new approach to improve system stability performance and increase computation efficiency simultaneously.

In this paper, we will consider the delay-dependent stability for systems with interval time-varying delay. By employing an innovative integral inequality to deal with the cross-product items in the derivative of Lyapunov functional, the less-conservative stability conditions are derived that use the least number of unknown LMI variables compared with existing stability criteria in the open literature, implying simultaneous improvement in stability performance and computational efficiency. The effectiveness of the proposed method has been demonstrated through a numerical example.

Problem formulation

Consider the following linear system with interval time-varying delay:

$$(1)\qquad \begin{aligned} &\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau(t)) \qquad t>0,\\ &x(t) = \phi(t), \qquad t\in[-\tau_{\max},0], \end{aligned}$$

where $x(t)\in\Re^n$ is the state variable; ϕ is a continuously differentiable initial function; A_0 and A_1 are known constant real matrices with appropriate dimensions. $\tau(t)$ is an interval time-varying state delay and satisfies $0\le\tau_{\min}\le\tau(t)\le\tau_{\max}$.

By defining $\tau_{med}=\frac{1}{2}(\tau_{\max}+\tau_{\min})$, where the subscript 'med' means the median of state delay, and $\delta=\frac{1}{2}(\tau_{\max}-\tau_{\min})$, where δ can be understood as the variation range of time-varying delay, then we obtain: $\tau(t)=\tau_{med}+\delta q(t)$,

where

$$q(t)=\begin{cases}\dfrac{2\tau(t)-(\tau_{\max}+\tau_{\min})}{\tau_{\max}-\tau_{\min}}, & \tau_{\max}>\tau_{\min}\\ 0 & \tau_{\max}=\tau_{\min}\end{cases}\quad |q(t)|\le 1.$$

So the time-varying delay is in the following interval:

$$(2)\qquad \tau(t)\in[\tau_{med}-\delta,\tau_{med}+\delta].$$

Remark 1: The delay described in (2) is not only an interval time-varying delay but also the uncertainty of time delay when modeling. In the traditional delay-dependent method, the time-varying delay is described as $\tau(t)\in[0\quad \bar{\tau}]$ and the maximum allowable upper delay bound can be obtained. When we define $\delta=\tau_{med}=\bar{\tau}/2$, then the traditional case for the time-varying delay is covered.

Remark2: In this paper there are no restrictions on the derivative of the time-varying delay, while traditional design methods require the derivative to be less than 1. So the proposed method can deal with fast time-varying delays, even when the time-varying delay is not differentiable, such as in Network Controlled Systems.

The objective of this work is to formulate some computable and less-conservative criteria to check the stability of the system (1) with interval time-varying delay. An LMI is proposed to obtain the sufficient condition of the stability which depend on τ_{med} and δ. The following improved lemma is derived from Jensen's integral inequality [6].It prevents a tighter bound to deal with cross-terms without ignoring any useful items.

Lemma 1: For any constant matrix $R \in \Re^{n \times n}$, $R = R^T > 0$, scalars $0 \le \tau_{\min} \le \tau(t) \le \tau_{\max}$, by defining $\tau_{med} = (\tau_{\max} + \tau_{\min})/2$, where the subscript 'med' means the median of time delays, and $\delta = (\tau_{\max} - \tau_{\min})/2$, where δ can be understood as the variation ranges of time-varying delays, and vector function $\dot{x}(t) : [-\tau_{\max}, -\tau_{\min}] \to \Re^n$ such that the following integration is well defined, it holds that

(3) $$-(h_2 - h_1)\int_{t-h_2}^{t-h_1} \dot{x}(s) R \dot{x}(s) ds \le \zeta^T(t) \Omega \zeta(t)$$

$$\begin{cases} \text{when } \tau_{\text{med}} \le \tau(t) \le \tau_{\max},\ h_2 = \tau_{\max},\ h_1 = \tau_{\text{med}} \\ \text{when } \tau_{\min} \le \tau(t) \le \tau_{\text{med}},\ h_2 = \tau_{\text{med}},\ h_1 = \tau_{\min} \\ \text{when } \tau_{\text{med}} \text{ is unknown},\ h_2 = \tau_{\max},\ h_1 = \tau_{\min} \end{cases}.$$

where

$$\zeta(t) = \begin{bmatrix} x(t-h_1) \\ x(t-\tau(t)) \\ x(t-h_2) \end{bmatrix},\ \Omega = \begin{bmatrix} -R & R & 0 \\ * & -2R & R \\ * & * & -R \end{bmatrix}.$$

Main results

Now we provide a delay-dependent stability criterion for system (1).

Theorem 1: Suppose that $\tau_{\min}, \tau_{\max}$ are known positive scalars, where $\tau_{\max} \ge \tau_{\min} \ge 0$, $\tau_{med} = (\tau_{\max} + \tau_{\min})/2$, $\delta = (\tau_{\max} - \tau_{\min})/2$. System (1) is asymptotically stable for any $\tau(t) \in [\tau_{\min}, \tau_{\max}]$, if there exist matrices with appropriate dimensions $P > 0$, $Q_i > 0 (i = 1,2)$, $R_j > 0 (j = 1,2)$ such that the following inequality holds:

(4) $$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ * & \Xi_{22} \end{bmatrix} < 0,$$

where

$$\Xi_{11} = \begin{bmatrix} \Sigma_{11} & R_1 & PA_1 & 0 \\ * & -Q_1 - R_1 - R_2 & R_2 & 0 \\ * & * & -2R_2 & R_2 \\ * & * & * & -Q_2 - R_2 \end{bmatrix},$$

$$\Xi_{12} = \begin{bmatrix} \tau_{med} A_0^T R_1 & \delta A_0^T R_2 \\ 0 & 0 \\ \tau_{med} A_1^T R_1 & \delta A_1^T R_2 \\ 0 & 0 \end{bmatrix},$$

$$\Xi_{22} = diag\{-R_1, -R_2\},$$

$$\Sigma_{11} = PA_0 + A_0^T P + Q_1 + Q_2 - R_1 - R_2.$$

Proof: We prove the theorem in two steps. First, we prove that the results shown in the theorem hold for $\tau_{med} \le \tau(t) \le \tau_{\max}$, and second, we prove that the results also hold for $\tau_{\min} \le \tau(t) \le \tau_{med}$. Then, we can infer that the theorem is true.

(1) when $\tau_{med} \le \tau(t) \le \tau_{\max}$.

Choose the Lyapunov-Krasovskii functional candidate as

(5) $$\begin{aligned} V(x_t) &= x^T(t) P x(t) + \int_{t-\tau_{med}}^{t} x^T(s) Q_1 x(s) ds \\ &+ \int_{t-\tau_{\max}}^{t} x^T(s) Q_2 x(s) ds \\ &+ \tau_{med} \int_{-\tau_{med}}^{0} ds \int_{t+s}^{t} \dot{x}^T(\theta) R_1 \dot{x}(\theta) d\theta \\ &+ \delta \int_{-\tau_{\max}}^{-\tau_{med}} ds \int_{t+s}^{t} \dot{x}^T(\theta) R_2 \dot{x}(\theta) d\theta. \end{aligned}$$

Taking time derivative of $V(x_t)$, we obtain

(6) $$\begin{aligned} \dot{V}(x_t) &= 2x^T(t) P \dot{x}(t) + x^T(t)[Q_1 + Q_2] x(t) \\ &- x^T(t-\tau_{med}) Q_1 x(t-\tau_{med}) \\ &- x^T(t-\tau_{\max}) Q_2 x(t-\tau_{\max}) \\ &+ x^T(t)[\tau_{med}^2 R_1 + \tau_{\max}^2 R_2] \dot{x}(t) \\ &- \tau_{med} \int_{t-\tau_{med}}^{t} \dot{x}^T(s) R_1 \dot{x}(s) ds \\ &- \delta \int_{t-\tau_{\max}}^{t-\tau_{med}} \dot{x}^T(s) R_2 \dot{x}(s) ds. \end{aligned}$$

Applying Lemma 1 of this paper and Jessen's inequality to deal with the cross-product items in (6), we have

(7) $$\begin{aligned} &-\tau_{med} \int_{t-\tau_{med}}^{t} \dot{x}^T(s) R_1 \dot{x}(s) ds \\ &\le \begin{bmatrix} x(t) \\ x(t-\tau_{med}) \end{bmatrix}^T \begin{bmatrix} -R_1 & R_1 \\ * & -R_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau_{med}) \end{bmatrix} \end{aligned}$$

(8) $$\begin{aligned} &-\delta \int_{t-\tau_{\max}}^{t-\tau_{med}} \dot{x}^T(s) R_2 \dot{x}(s) ds \\ &\le \begin{bmatrix} x(t-\tau_{med}) \\ x(t-\tau(t)) \\ x(t-\tau_{\max}) \end{bmatrix}^T \begin{bmatrix} -R_2 & R_2 & 0 \\ * & -2R_2 & R_2 \\ * & * & -R_2 \end{bmatrix} \begin{bmatrix} x(t-\tau_{med}) \\ x(t-\tau(t)) \\ x(t-\tau_{\max}) \end{bmatrix} \end{aligned}$$

Substituting (7,8) into (6) yields

(9) $$\dot{V}(x_t) \le \xi^T(t)[\Xi_{11} - \Xi_{12}\Xi_{22}^{-1}\Xi_{12}^T]\xi(t).$$

where

$\xi^T(t) = [x^T(t), x^T(t-\tau_{med}), x^T(t-\tau(t)), x^T(t-\tau_{\max})]$ and $\Xi_{11}, \Xi_{12}, \Xi_{22}$ are defined in Theorem 1.

Using Schur complements, we can see that (4) is equivalent to $\Xi_{11} - \Xi_{12}\Xi_{22}^{-1}\Xi_{12}^T < 0$, so if (4) holds, that is $\dot{V}(x_t) < 0$. Hence the system is asymptotically stable.

(2) when $\tau_{\min} \le \tau(t) \le \tau_{med}$

For this case, we choose the Lyapunov-Krasovskii functional candidate as

$$V(x_t) = x^T(t)Px(t) + \int_{t-\tau_{med}}^{t} x^T(s)Q_1x(s)ds$$
$$+\int_{t-\tau_{\min}}^{t} x^T(s)Q_2x(s)ds \tag{10}$$
$$+\tau_{med}\int_{-\tau_{med}}^{0} ds\int_{t+s}^{t}\dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta$$
$$+\delta\int_{-\tau_{med}}^{-\tau_{\min}} ds\int_{t+s}^{t}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta.$$

The proof process is similar to that for the case of $\tau_{med} \le \tau(t) \le \tau_{\max}$, except that $\xi^T(t)$ is chosen to be $\xi^T(t) = [x^T(t), x^T(t-\tau_{med}), x^T(t-\tau(t)), x^T(t-\tau_{\min})]$. Hence, condition (4) can be obtained. This completes the proof.

Numerical Examples

In this section, we use an example in Shao (2008) to show the reduced conservatism of our results.

Consider system (1) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

The allowable upper bound $\tau_{\max}$ of the delay are shown with given $\tau_{\min}$ in Table 1.

As shown in the table, the new stability criterion in this paper has advantages over the previous ones in the sense that the computed allowable upper bound of time delay is larger. It is worth pointing out that the developed stability criterion is significantly simplified because neither any model transformation nor free weighting matrices are employed in the theoretical derivation.

Table 1. The parameters of the sensor

Method	Jiang and Han	He et al.	Shao	Theorem1
$\tau_{\min}$	$\tau_{\max}$	$\tau_{\max}$	$\tau_{\max}$	$\tau_{\max}$
0	0.67	0.77		1.218
0.3	0.91	0.9431	0.9806	1.424
0.5	1.07	1.0991	1.1325	1.568
0.8	1.33	1.3476	1.3733	1.794
1.0	1.50	1.5187	1.5401	1.948
2.0	2.39	2.4	2.41	2.766

Conclusion

In this paper, the stability problem has been investigated for systems with a time delay varies in an interval. Simplified and improved delay-dependent stability criterion has been established by using a new Lyapunov–Krasovskii functional and an innovative integral inequality. The numerical results seem to suggest that the proposed methods may improve the results in some existing papers.

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