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NEW NONLINEAR TRACKING DIFFERENTIAL-ESTIMATORS: THEORY AND PRACTICE

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Abstract. *Differential estimation is required for many fields. This paper proposes novel nonlinear tracking differential-estimators to solve the problem of phase lag of traditional approach. In this study, the conventional structure of nonlinear tracking differentiator is expanded. First, the lowest-order design is proposed, which improves the rapidity by reducing one equation compared with the traditional method. Moreover, the idea of feedforward is introduced into constructing the equations set of the differential-estimator to improve the rapidity further. The numerical examples are given, and the practical application experiments are implemented in a servo system. According to the results, the proposals have smaller phase lag, and achieve better practical effect.*
Keywords: Nonlinear tracking differentiator, Rapidity, Phase lag, Feedforward

1. **Introduction.** Recently, the necessity to estimate the time-derivative of signals arises in many fields [1, 2]. For instance, one needs to obtain a velocity signal rapidly and accurately from a position measurement in a robot and other servo systems, or to calculate the rate of a reactant temperature change in chemical industry. In practice, the simplest method is the backward difference

$$\hat{\dot{v}}(k) = \frac{v(k) - v(k-1)}{T_s}, \tag{1}$$

where $\hat{\dot{v}}(k)$ is the differential estimation of a measured signal $v(k)$ at the kth sampling instant. T_s is a sampling period. However, this normal difference amplifies the noise in the signal, and does not provide satisfactory estimation.

Therefore, many researchers have tried to develop differentiators to estimate the differential of a signal. Levant proposed a differentiator via sliding mode technique [3]. This method obtains the derivative of an input signal, and inhibits the influence of signal noise in some degree. However, the information that one needs to know on the signal is an upper bound for Lipschitz's constant of the derivative of the signal. Besides, the chattering phenomenon is inevitable [4]. J. Han et al. proposed the dynamics structure of a nonlinear tracking differentiator (NTD) [5] which successfully realized the estimation of differential. The NTD method benefits from the "tracking", and estimates the differential through integral actions. This approach has been studied widely in both theory and practice since it came into being [3, 5-13].

Previous researches usually focus on the design of the nonlinear function in a NTD to get faster differential estimation. However, there is always serious phase lag of the conventional NTD no matter how to design the nonlinear function. Because of the phase lag, the accuracy of differential estimation is degraded, which leads to the degradation of the practical effect of the conventional NTD, especially in a high-speed and high-accuracy servo system.

In this paper, we try to develop new differential-estimators to decrease the phase lag of differential estimation and improve the practical effect in real system by changing the structure of the conventional NTD. Two schemes, the lowest-order nonlinear tracking differential-estimator (LNTD) and the feedforward-constructing nonlinear tracking differential-estimator (FNTD), are proposed. These methods maintain the profit of "tracking" of NTD; however, they are simpler and faster than the traditional method. A trade-off is achieved between fast estimation and noise tolerance: the phase lag of differential estimation is effectively decreased, and the noise is not amplified seriously. Numerical examples are given in this paper. Moreover, the practical application is researched for high-speed and high-accuracy servo by experiments, which verify the effectiveness.

This paper is organized as follows. The problem is described in Section 2. Then, the LNTD and the FNTD are proposed in Section 3 and Section 4, respectively. The numerical examples are given in Section 5. The practical application research is carried out in Section 6. Finally, the paper is concluded in Section 7.

2. Problem Formulation.

In this section, the conventional design objective is given first. Then, the power function based NTD is introduced as an example of the conventional design. Finally, the problem of the conventional design is described.

2.1. Design objective of conventional NTD.

Tracking differentiator is such a system [5]: for a given input signal $v(t)$, the system provides two signals $x_1(t)$ and $x_2(t)$ such that the two signals track $v(t)$ and its derivative $\dot{v}(t)$ [12], i.e.,

$$x_1(t) \to v(t), \;\; x_2(t) \to \dot{v}(t). \tag{2}$$

In the previous work, researchers have given a general structure of NTD as (3)

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = R^2 f\left[x_1(t) - v(t), \dfrac{x_2(t)}{R}\right], \end{cases} \tag{3}$$

where $f(\bullet, \bullet)$ is a nonlinear function; R ($R > 0$) is a real constant. The traditional design of NTD is to select appropriate $f(\bullet, \bullet)$ to make the system (3) asymptotically stable. Meanwhile, the system should make the solution $x_1(t)$ satisfy

$$\lim_{R \to +\infty} \int_0^T |x_1(t) - v(t)| \, dt = 0, \tag{4}$$

for any real number $T > 0$, which means the state variable $x_1(t)$ converges to the input signal $v(t)$ in mean. Then, $x_2(t)$ converges to the generalized derivative of $v(t)$. The structure of the NTD is illustrated in Figure 1 which is a nonlinear feedback system.

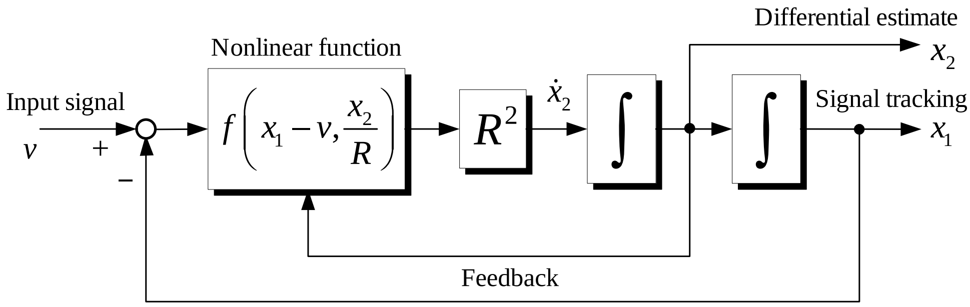

FIGURE 1. The structure of the conventional NTD

2.2. **Power function based NTD.** There is a satisfactory design [6] based on the power function

$$f(z_1, z_2) = -\alpha_1 \left[(\beta z_1)^{\frac{p}{q}} + z_1\right] - \alpha_2 \left[(\beta z_2)^{\frac{p}{q}} + z_2\right], \tag{5}$$

where z_1, z_2 are the arguments of the function; constants α_1, α_2 are greater than zero, and β is greater than 1; constants p and q ($p > q > 0$) are both odd numbers.

This function is composed of a linear part and a nonlinear part to balance the contradiction between the rapidity and the stationarity of NTD. For this function, the nonlinear part plays a major role when $|\beta z_i| \gg 1$, $i = 1, 2$. The linear part plays a major role when $|\beta z_i| \ll 1$, $i = 1, 2$. The β is a weighting. The p and q determine the shape of the nonlinear part. Bigger β and p/q mean faster convergence speed of the function.

Based on the function (5), the nonlinear tracking differentiator is obtained as (6).

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = R^2 \left\{ -\alpha_1 \left\{ \beta^{\frac{p}{q}} \left[x_1(t) - v(t)\right]^{\frac{p}{q}} + \left[x_1(t) - v(t)\right] \right\} - \alpha_2 \left\{ \left[\dfrac{\beta x_2(t)}{R}\right]^{\frac{p}{q}} + \dfrac{x_2(t)}{R} \right\} \right\} \end{cases} \tag{6}$$

Such as the power function based NTD, conventional designs use the form (3) and choose different nonlinear function. In practice, a conventional NTD outputs x_1 and x_2, the tracking of the input signal and the estimation of the differential, via two integral actions. According to control theory, an integral action leads to 90 degrees phase lag at the most. Therefore, there is always phase lag fundamentally caused by the integral actions in conventional NTD.

3. **Lowest-Order NTD.** In this section, the conventional power function based NTD (6) is modified. The phase lag can be reduced by decreasing the number of integral actions in solving the equations, since it is caused by the integral fundamentally. The new scheme is proposed, which uses only one variable to realize the target of differential estimation. To propose this differentiator, following lemmas are given first.

Lemma 3.1. *For the nonlinear Equation (7) regarding the variable $z(t)$,*

$$\dot{z}(t) = -\left[\beta z(t)\right]^{\frac{p}{q}} - z(t), \tag{7}$$

if

(1) constant β is greater than 1,
(2) p, q ($p > q > 0$) are both odd numbers,

then the solution of system (7) is globally asymptotically stable to zero.

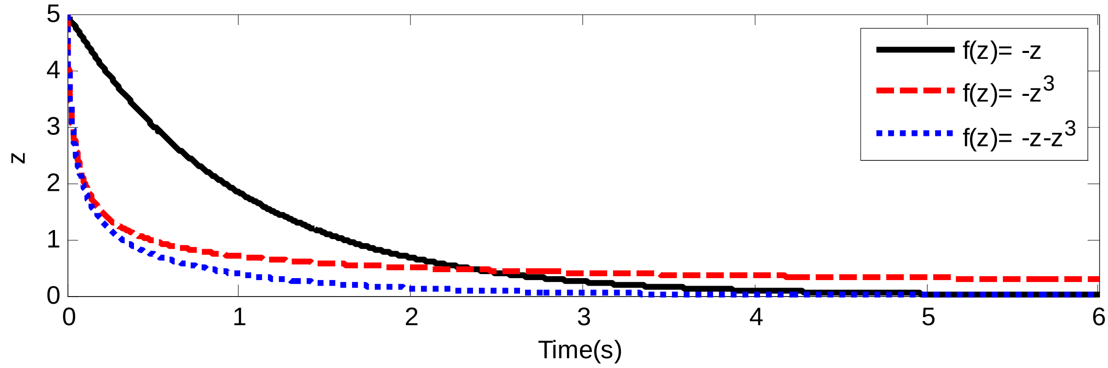

FIGURE 2. Convergence of the power function and the linear function

Proof: Select the Lyapunov candidate

$$V(z) = \frac{1}{2}z^2(t). \tag{8}$$

It is reasonable that $V(z) \geq 0$ and $V(0) = 0$. The total derivative of $V(z)$ with respect to time t along the trajectory of system (7) is

$$\frac{dV(z)}{dt} = \dot{z}(t)z(t) = -\beta^{\frac{p}{q}} z(t)^{\frac{p+q}{q}} - z^2(t). \tag{9}$$

Because p and q are odd, $\frac{dV(z)}{dt}$ is not greater than zero. Therefore, $V(z)$ is a Lyapunov function. In the neighborhood of zero, only $z(t) = 0$ makes $\frac{dV(z)}{dt} = 0$. According to Barbashin-Krasovskii-LaSalle Theorem [15], the solution of the system (7) is globally asymptotically stable to zero. There is $z(t) \to 0$ when $t \to +\infty$. □

Remark 3.1. *The right part of (7) is also composed of a linear function and a power function regarding the variable z. Figure 2 shows the difference on the convergence of the linear part and the nonlinear part using an example. According to the figure, the power function has fast speed of convergence when the variable is far away from the equilibrium point. However, it converges quite slowly when the variable is near the equilibrium point. The linear function has stable and faster convergence near the equilibrium point, however, slowly converge when the variable is far away from the equilibrium. Therefore, the function used in (7) has fast convergence whatever the value of the variable z is.*

Then, we will give the convergence of the differential-estimator which is constructed by a first-order equation. The convergence of tracking differentiator is seen as an independent problem which was presented the first time in [5], and re-proved in [10]. However, it is recently pointed out that the previous proof is incomplete, and the mistake is finally removed by B. Guo and Z. Zhao [16]. The latest result is shown in Appendix A. We directly use this result to support the following lemma which can be seen as a corollary of Theorem 2.1 in [16].

Lemma 3.2. *Suppose that the equilibrium point* 0 *of the following system is globally asymptotically stable:*

$$\dot{z}(t) = f[z(t)], \quad z(0) = z_0, \tag{10}$$

where

z_0 *is any initial value,*
$f(\bullet)$ *is a locally Lipschitz continuous function,* $f(0) = 0$.

If a signal $v(t)$ is differentiable and $\sup_{t\in[0,+\infty)} |\dot{v}(t)| < +\infty$, then the solution $x_R(t)$ of the system

$$\dot{x}_R(t) = Rf[x_R(t) - v(t)], \quad x_R(0) = x_{R0} \tag{11}$$

is convergent in the sense that: for every $a > 0$, $x_R(t)$ is uniformly convergent to $v(t)$ on $t \in [a, +\infty)$ as $R \to +\infty$, where x_{R0} is any initial value.

Proof: Suppose $h = Rt$ and $y_R(h) = x_R(t) - v(t)$. Then

$$\frac{dy_R(h)}{dt} = R\dot{y}_R(h) = \dot{x}_R(t) - \dot{v}(t) = Rf[x_R(t) - v(t)] - \dot{v}(t). \tag{12}$$

Therefore, we have

$$\dot{y}_R(h) = f[x_R(t) - v(t)] - \frac{\dot{v}(t)}{R} = f[y_R(h)] - \frac{\dot{v}(t)}{R}. \tag{13}$$

By changing the argument, (13) can be written as

$$\dot{y}_R(t) = f[y_R(t)] + g_R(t) \tag{14}$$

where $g_R(t) = -\frac{\dot{v}(t/R)}{R}$.

Therefore, $y_R(t)$ is a solution to the perturbed system of (10). Since $f(\bullet)$ is locally Lipschitz continuous and system (10) is globally asymptotically stable to zero, the exitance of Lyapunov function of system (10) is confirmed. The convergence of system (11) is not difficult to prove, which can be completed according to the line of Lemma A.1 in the appendix. □

Combining Lemma 3.1 and Lemma 3.2, the following theorem is obtained.

Theorem 3.1. *If it is satisfied that*

(1) constant β is greater than 1,

(2) constants p and q $(p > q > 0)$ are both odd numbers,

(3) the signal $v(t)$ is differentiable and $\sup_{t\in[0,+\infty)} |\dot{v}(t)| < +\infty$, then the solution of following system:

$$\dot{x}_R(t) = R\left\{-\beta^{\frac{p}{q}}\left[x_R(t) - v(t)\right]^{\frac{p}{q}} - \left[x_R(t) - v(t)\right]\right\}, \quad x_R(0) = x_{R0} \tag{15}$$

is convergent in the sense that: for every $a > 0$, $x_R(t)$ is uniformly convergent to $v(t)$ on $t \in [a, +\infty)$ as $R \to +\infty$, where x_{R0} is any initial value.

Equation (15) is the LNTD, a first-order differential equation. Since $x_R(t)$ is convergent to $v(t)$, $\dot{x}_R(t)$ is the estimation of $\dot{v}(t)$. This differential-estimator uses only one variable to estimate the differential of the input signal by solving the differential equation: $\dot{x}_R(t)$ is calculated using old x_R, then the $x_R(t)$ is obtained by integrating the $\dot{x}_R(t)$ as shown in Figure 3. This proposal decreases one integral action compared with the conventional NTD, which reduces the phase lag of differential estimation. Moreover, two parameters are reduced, which makes the parameters adjustment easier.

Remark 3.2. *This proposed method also maintains noise-tolerance in some degree. In the normal difference method, the noise in the input signal is amplified by $1/T_s$ times and outputted into the differential estimation. In modern digital control device, T_s is usually programmed as $0.001s \sim 0.0001s$ to guarantee the control performance. Therefore, the noise is amplified by $1000 \sim 10000$ times. However, the noise in the proposed method is very small due to the effect of integral in Figure 3. The differential estimation $\dot{x}_R$ is obtained from x_R that has tiny noise. It is to say that using indirect integral calculation to estimate the differential leads to better noise-tolerance than direct difference method. Although, the traditional NTD has another variable and another integral action that filter*

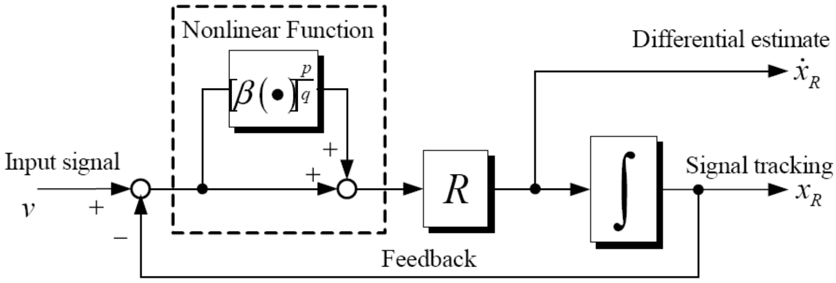

FIGURE 3. The structure of the LNTD

the noise better, the integral causes more serious phase lag. Actually, the rapidity and the noise tolerance are conflicting. The proposed method is a trade-off: it has better dynamics than traditional NTD structure, and maintains the noise-tolerance. In fact, for a high-speed and high-accuracy servo system, smaller phase lag in signal measurement is more beneficial to expend the bandwidth of the system.

4. **Feedforward-Constructing NTD.** For a tracking differential-estimator, the realization of signal tracking and differential estimation benefits from the feedback of the estimation error. In control theory, feedforward is also important in the aspect of improving system dynamic property. Therefore, this section proposes a new structure using feedforward. In order to propose this method, the following lemmas are needed.

Lemma 4.1. *There is a nonlinear differential equations set regarding the variables* $z_1(t)$ *and* $z_2(t)$

$$\begin{cases} \dot{z}_1(t) = -\left[\beta z_1(t)\right]^{\frac{p}{q}} - z_1(t) \\ \dot{z}_2(t) = -\left\{\left[\beta z_2(t)\right]^{\frac{p}{q}} + z_2(t)\right\} - \alpha\left\{\left[\beta z_1(t)\right]^{\frac{p}{q}} + z_1(t)\right\}, \end{cases} \tag{16}$$

where

constant β *is greater than 1;*
constants p *and* q $(p > q > 0)$ *are both odd;*
α $(0 < \alpha < 2)$ *is also a constant.*

Then, the solution (z_1, z_2) *of the system (16) is globally asymptotically stable to* $(0, 0)$.

Proof: Choose the Lyapunov candidate

$$V(z_1, z_2) = \frac{q}{p+q}\beta^{\frac{p}{q}} z_1^{\frac{p+q}{q}}(t) + \frac{1}{2}z_1^2(t) + \frac{q}{p+q}\beta^{\frac{p}{q}} z_2^{\frac{p+q}{q}}(t) + \frac{1}{2}z_2^2(t). \tag{17}$$

Then, $V(z_1, z_2) \geq 0$ and $V(0, 0) = 0$ are tenable because p, q are odd. The total derivative of $V(z_1, z_2)$ with respect to time t along the system (16) is

$$\begin{aligned} \frac{dV(z_1, z_2)}{dt} &= \beta^{\frac{p}{q}} z_1^{\frac{p}{q}}(t)\dot{z}_1(t) + \dot{z}_1(t)z_1(t) + \beta^{\frac{p}{q}} z_2^{\frac{p}{q}}(t)\dot{z}_2(t) + \dot{z}_2(t)z_2(t) \\ &= -\dot{z}_1^2(t) - \dot{z}_2^2(t) + \alpha\dot{z}_1(t)\dot{z}_2(t) \\ &= -\frac{(2-\alpha)}{2}\dot{z}_1^2(t) - \frac{\alpha}{2}\dot{z}_1^2(t) - \frac{(2-\alpha)}{2}\dot{z}_2^2(t) - \frac{\alpha}{2}\dot{z}_2^2(t) + \alpha\dot{z}_1(t)\dot{z}_2(t) \\ &= -\frac{(2-\alpha)}{2}\left[\dot{z}_1^2(t) + \dot{z}_2^2(t)\right] - \frac{\alpha}{2}\left[\dot{z}_1(t) - \dot{z}_2(t)\right]^2 \leq 0. \end{aligned} \tag{18}$$

Therefore, $V(z_1, z_2)$ is a Lyapunov function. To satisfy $\frac{dV(z_1,\ z_2)}{dt} = 0$, there should be

$$\begin{cases} \dot{z}_1(t) = 0 \\ \dot{z}_2(t) = 0 \\ \dot{z}_1(t) - \dot{z}_2(t) = 0 \end{cases} \tag{19}$$

Notice that only $\dot{z}_1(t) = 0$ and $\dot{z}_2(t) = 0$ satisfy the three equations in (19) at the same time. And only when $(z_1,\ z_2) = (0, 0)$, $\dot{z}_1(t)$ and $\dot{z}_2(t)$ are both zeros at the same time. Therefore, only $(0, 0)$ makes $dV(z_1,\ z_2)/dt = 0$. Also according to Barbashin-Krasovskii-LaSalle Theorem, the solution of (16) is globally asymptotically stable to $(0, 0)$. □

Then, we also give following Lemma 4.2 to show the convergence of the differential-estimator which is constructed using feedforward.

Lemma 4.2. *Suppose that the equilibrium point* $(0, 0)$ *of the following system is globally asymptotically stable:*

$$\begin{cases} \dot{z}_1(t) = f[z_1(t)], \quad z_1(0) = z_{10} \\ \dot{z}_2(t) = f[z_2(t)] + \alpha f[z_1(t)], \quad z_2(0) = z_{20}, \end{cases} \tag{20}$$

where

(z_{10}, z_{20}) *is any initial value;*
R *and* α *are constants;*
$f(\bullet)$ *is a locally Lipschitz continuous function,* $f(0) = 0$.

If the signal $v(t)$ *is differentiable and* $\sup_{t\in[0,+\infty)} |\dot{v}(t)| < +\infty$, *then the solution* $x_{R2}(t)$ *of the system*

$$\begin{cases} \dot{x}_{1R}(t) = Rf[x_{1R}(t) - v(t)], \quad x_{1R}(0) = x_{10} \\ \dot{x}_{2R}(t) = R\left\{f[x_{2R}(t) - v(t)] + \alpha f[x_{1R}(t) - v(t)]\right\}, \quad x_{2R}(0) = x_{20}, \end{cases} \tag{21}$$

is convergent in the sense that: for every $a > 0$, $x_{2R}(t)$ *is uniformly convergent to* $v(t)$ *on* $t \in [a, +\infty)$ *as* $R \to +\infty$, *where* (x_{10}, x_{20}) *is any initial value.*

Proof: Let $h = Rt$, and

$$\begin{cases} y_{1R}(h) = x_{1R}(t) - v(t) \\ y_{2R}(h) = x_{2R}(t) - v(t). \end{cases} \tag{22}$$

Then

$$\begin{cases} \dot{y}_{1R}(h) = f[y_{1R}(h)] - \dfrac{\dot{v}(t)}{R} \\ \dot{y}_{2R}(h) = f[y_{2R}(h)] + \alpha f[y_{1R}(h)] - \dfrac{\dot{v}(t)}{R} \end{cases} \tag{23}$$

Therefore, $\mathbf{y}_R = (y_{1R}, y_{2R})^T$ is a solution to the system

$$\dot{\mathbf{y}}_R(t) = F[\mathbf{y}_R(t)] + \mathbf{g}_R(t), \tag{24}$$

where

$$F[\mathbf{y}_R(t)] = \left\{f[y_{1R}(t)],\ f[y_{2R}(t)] + \alpha f[y_{1R}(t)]\right\}^T, \tag{25}$$

$$\mathbf{g}_R(t) = \left[-\frac{\dot{v}\left(\frac{t}{R}\right)}{R},\ \ -\frac{\dot{v}\left(\frac{t}{R}\right)}{R}\right]^T. \tag{26}$$

If $\mathbf{z} = (z_1, z_2)^T$ is a solution to the system (20), then Equation (20) can be rewritten as

$$\dot{\mathbf{z}} = F[\mathbf{z}(t)]. \tag{27}$$

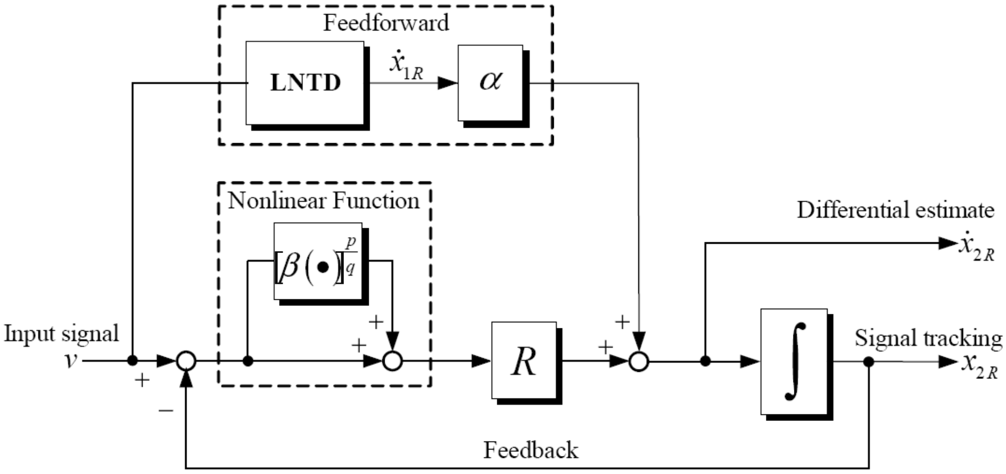

FIGURE 4. The structure of the FNTD

Therefore, (24) is a perturbed system of (27). It is not difficult to verify the convergence of system (21) according to Lemma A.1 in the appendix. □

Then, combining Lemma 4.1 and Lemma 4.2, the following theorem is obtained.

Theorem 4.1. *If it is satisfied that*

(1) $0 < \alpha < 2$*,*

(2) β *is greater than 1,*

(3) constants p *and* q $(p > q > 0)$ *are both odd numbers,*

(4) the signal $v(t)$ *is differentiable and* $\sup_{t\in[0,+\infty)} |\dot{v}(t)| < +\infty$*, then the solution* $x_{R2}(t)$ *of the system*

$$\begin{cases} \dot{x}_{1R}(t) = R\left\{-\beta^{\frac{p}{q}}\left[x_{1R}(t) - v(t)\right]^{\frac{p}{q}} - \left[x_{1R}(t) - v(t)\right]\right\}, & x_{1R}(0) = x_{10} \\ \dot{x}_{2R}(t) = R\left\{-\beta^{\frac{p}{q}}\left[x_{2R}(t) - v(t)\right]^{\frac{p}{q}} - \left[x_{2R}(t) - v(t)\right]\right\} + \alpha\dot{x}_{1R}(t), & x_{2R}(0) = x_{20} \end{cases} \tag{28}$$

is convergent in the sense that: for every $a > 0$*,* $x_{2R}(t)$ *is uniformly convergent to* $v(t)$ *on* $t \in [a, +\infty)$ *as* $R \to +\infty$*, where* (x_{10}, x_{20}) *is any initial value.*

Equation (28) is the FNTD. $\dot{x}_{2R}(t)$ is used as the differential estimation of the input signal $v(t)$. This differential-estimator is composed of two LNTDs as shown in Figure 4. The output of the first LNTD is injected into the second one, which realizes the effect of feedforward. Here, α is the gain of feedforward. According to control theory, bigger α allows smaller phase lag. However, excessive α can also lead to an overshoot. Therefore, the design of α should trade off the rapidity and the overshoot. In practice, α can be designed between 0 to 0.5 to decrease phase lag effectively without causing large overshoot. This differential-estimator responds faster, which is more suitable for the requirement of higher accuracy of the differential estimation.

5. **Numerical Examples.** In order to test the performance of the proposed differential-estimators, the numerical examples are given using sine signals as the inputs with the magnitude of 0.005 m and the frequencies from 0.2 Hz to 10 Hz.

In this section, the conventional NTD (6) is also implemented for comparison. The parameters are adjusted by the method of trial-and-error as shown in Table 1. These parameters make the status of every approach similar to each other. Euler method is employed to solve the differential equations numerically. The integration step is 0.1 ms.

Table 1. The parameters of the conventional and the proposed methods

Parameters	Conventional NTD Equation (6)	LNTD Equation (15)	FNTD Equation (28)
R	500.0	500.0	500.0
α_1	1.0	–	–
α_2	2.0	–	–
α	–	–	0.35
β	30.0	30.0	30.0
p	3	3	3
q	1	1	1

Table 2. The phase lags of each method

Frequency (Hz)	Conventional NTD (deg)	LNTD (deg)	FNTD (deg)
0.2	–0.2772	–0.1404	–0.0900
0.4	–0.5543	–0.2808	–0.1800
0.6	–0.8315	–0.4212	–0.2700
0.8	–1.1087	–0.5616	–0.3600
1.0	–1.3860	–0.7020	–0.4500
2.0	–2.7708	–1.4036	–0.9001
4.0	–5.5160	–2.8021	–1.8002
6.0	–8.2210	–4.1909	–2.7002
8.0	–10.8955	–5.5680	–3.5991
10.0	–13.4906	–6.9307	–4.5009

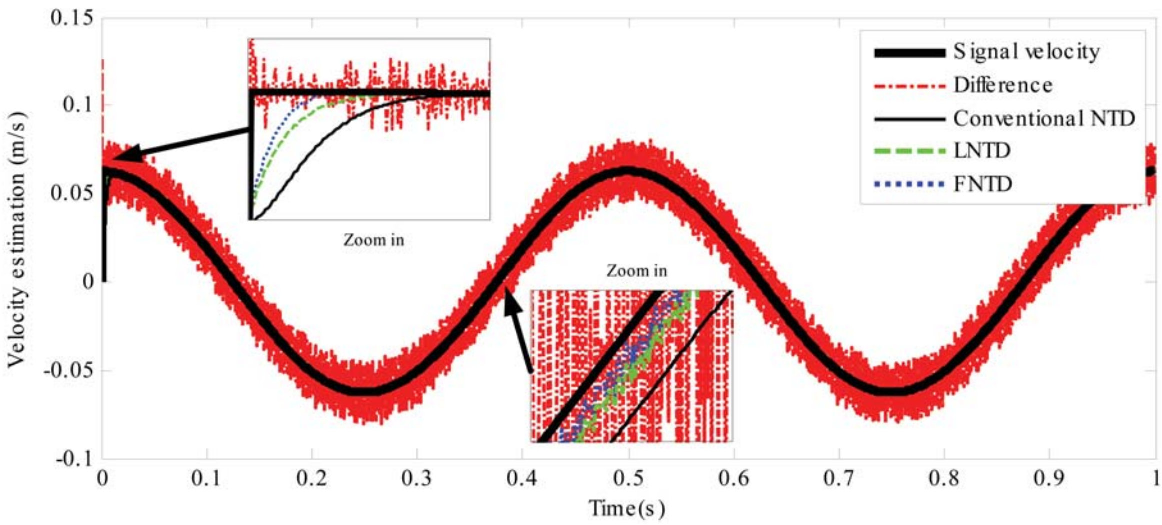

Figure 5. A numerical example with noise

Table 2 illustrates the least square analysis of the phase lag of each method. The phase lags of the proposed LNTD and FNTD are significantly smaller than the conventional method. The FNTD achieves the smallest phase lag. The improvements effectively reduce the phase lag for differential estimation.

Figure 5 shows the situation when noise exists in the input signal. The input is a sinusoid with the magnitude of 0.005 m and the frequency of 2 Hz. A random noise between

± 1 μm that is ten times of the resolution of the sensor used in the next section is added into the input signal. For comparison, normal difference method is also implemented here.

According to the results, there is serious noise in the estimation of the normal difference method that is difficult to use in practical engineering. The conventional NTD has smallest noise; however, there also exists the largest phase lag which is adverse to implement some robust controller in practical engineering. The noise in the proposed methods is a little bigger than traditional NTD, but is limited in in a small and certain extent and obviously smaller than the normal difference method. Meanwhile, the estimation of the proposed methods is faster, and closer to the real value comparing with the conventional NTD.

6. **Practical Application Research.** The application of the proposed methods is studied for a high-speed and high-accuracy servo control in this section. The control plant is a linear motor (GMC Hillstone) servo system as shown in Figure 6. The position sensor is a photoelectric encoder (RENISHAW RCH24Y15A30A) with the resolution of 0.1 μm.

In this system, the algorithms of velocity loop disturbance observer (DOB) [17], the velocity feedback, the velocity feedforward, and the acceleration feedforward are realized by the traditional NTD and the proposed differential-estimators respectively with the parameters in Table 1. The whole control strategy is shown in Figure 7.

The control algorithms are realized in the Linux-RTAI real time system. The sampling interval is 0.1 ms. The experiments are composed of two cases: the state of low speed and the state of high speed. The position command is correspondingly set as $0.001[\cos(0.4\pi t) - 1]$ (m) and $0.001[\cos(20\pi t) - 1]$ (m). The experimental parameters are shown in Table 3.

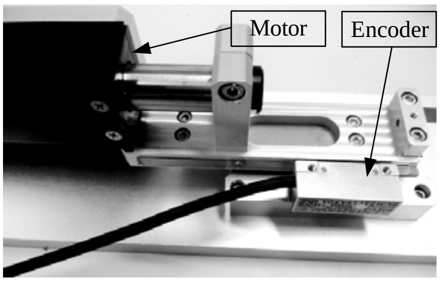

Figure 6. Experimental setup

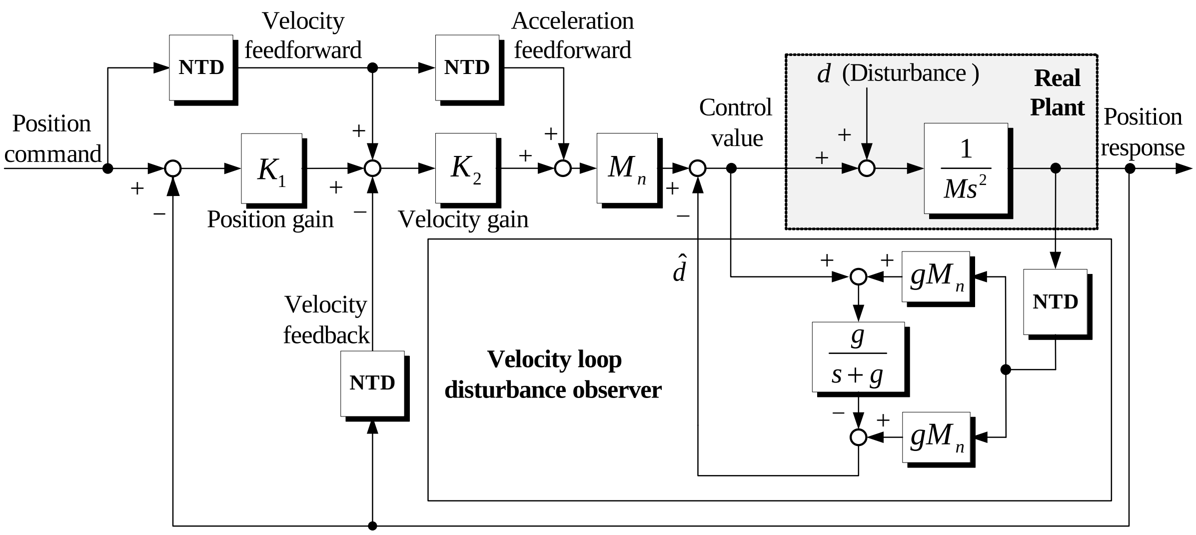

Figure 7. Application of NTD in high accuracy servo system

Table 3. The experimental parameters

Parameters	Marks	Value
Position gain	K_1	10.0 s^{-2}
Velocity gain	K_2	40.0 s^{-1}
DOB cutoff frequency	g	500.0 rad/s
Nominal mass	M_n	0.5 kg

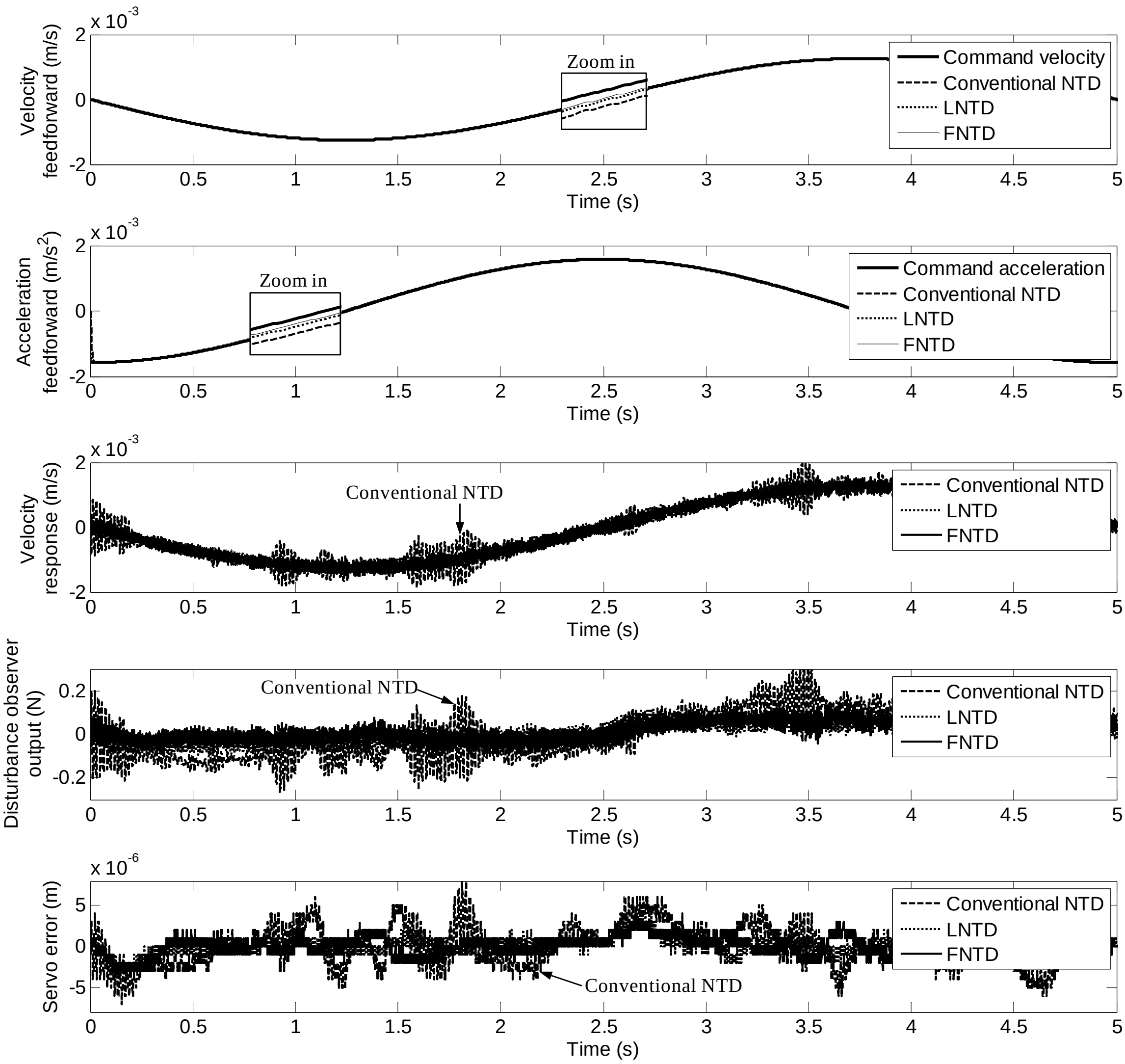

Figure 8. The results of application experiment

The results are illustrated in Figure 8 and Figure 9. Each one is composed of five sub-figures: velocity of the position command, acceleration of the position command, velocity response of the servo system, disturbance observer output, and the servo error.

In the state of low speed, according to the first two sub-figures of Figure 8, each method successfully achieves the estimations of the velocity and the acceleration of the position command. However, the proposed LNTD and FNTD have smaller phase lag than the conventional NTD. In which, the FNTD provides the fastest differential estimation.

The third sub-figure of Figure 8 illustrates the velocity response of the real servo system. The quality of the differential estimation is guaranteed effectively. However, there is

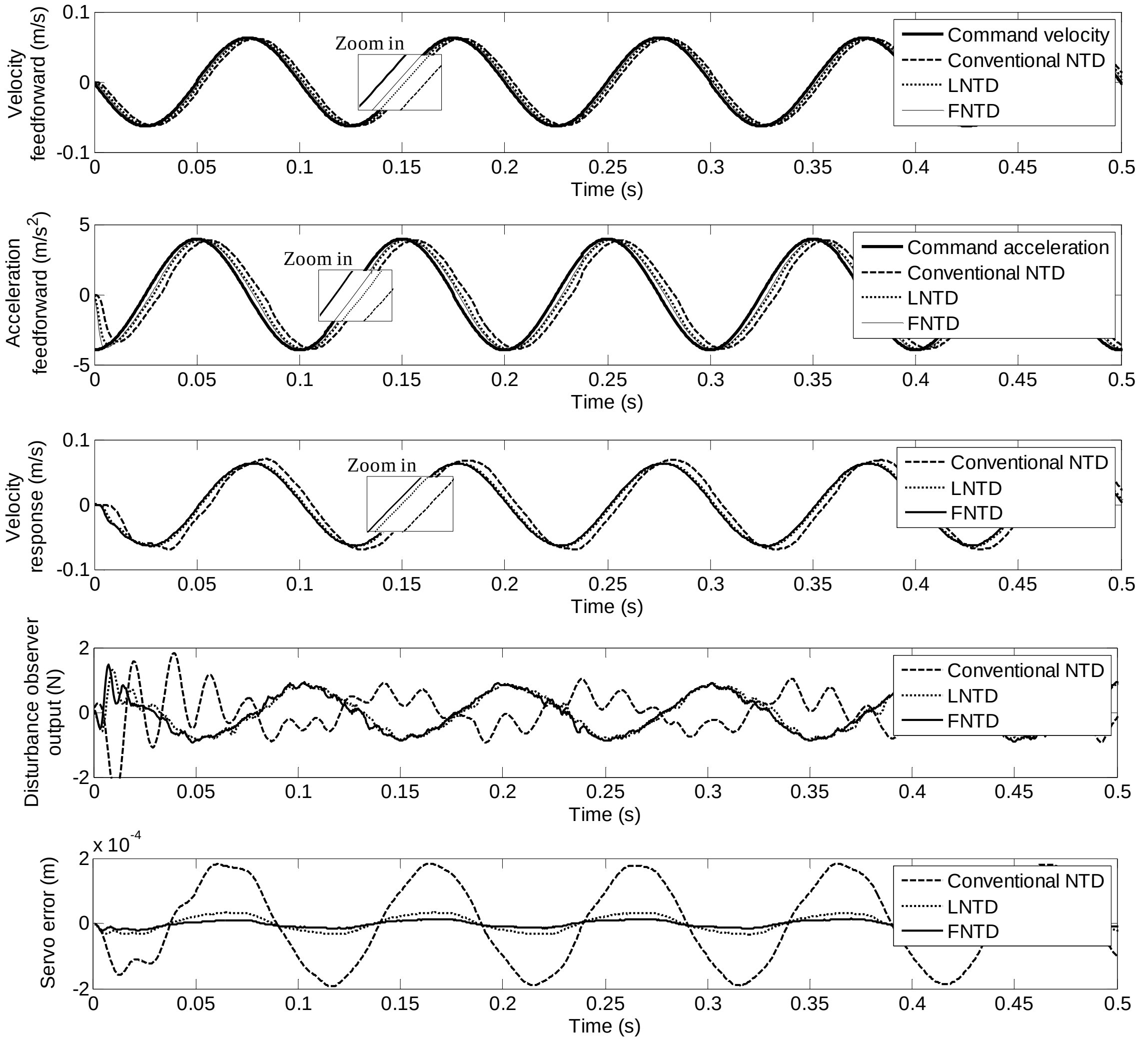

FIGURE 9. The results of application experiment

obvious oscillation in the response using the conventional NTD method. This is caused by the oscillation in the disturbance compensation value of the disturbance observer.

The output of the disturbance observer is illustrated in the fourth sub-figure of Figure 8. Due to the phase lag in the velocity estimation, the observed disturbance is fluctuant using the conventional NTD. This fluctuation does not only influence the velocity of the servo system, but also degrades the accuracy. According to the fifth sub-figure of Figure 8, the servo error is smaller when the proposed methods are used.

In the state of high speed, the responses are similar to the situation of low speed according to Figure 9. The rapidity and effectiveness in practice of the proposed method is more obvious. Especially, according to the third sub-figure of Figure 9, the noise is hardly to watch for the proposals. This profits from two aspects. On the one hand, the device is high-accuracy mechatronics system which has big signal-to-noise ratio in the state of high speed. On the other hand, the proposed methods still maintain noise-tolerance in some degree due to the integral action.

According to the last two sub-figures of Figure 9, the phase lag and the oscillation in the output of the DOB is more obvious when the conventional NTD is used. The DOB method is a highly robust method against disturbances and parameter uncertainties [18]. The phase lag in velocity estimation degrades the stability and the performance of DOB.

The phase lag and the oscillation are effectively weakened when the new approaches are used. Correspondingly, the servo error is 1.84×10^{-4} (m) at the most when conventional NTD is used. However, it is decreased to 0.33×10^{-4} (m) when we use the LNTD. Moreover, smaller servo error of 0.14×10^{-4} (m) is achieved by employing the FNTD. The performance of the servo system is effectively improved.

7. **Conclusions.** This paper considers the problem of the phase lag of nonlinear tracking differentiator method, and proposes two kinds of new structures. First, the lowest-order nonlinear tracking differential-estimator (LNTD) is proposed. This proposal decreases one differential equation and two parameters compared with the traditional method, which achieves smaller phase lag and easier parameters adjustment.

Then, the idea of feedforward is introduced into the design of differential-estimator. The feedforward-constructing nonlinear tracking differential-estimator (FNTD) is proposed by composing two LNTDs. This method improves the rapidity of differential estimation further. Faster and more accurate differential estimation is achieved, which satisfies higher requirement of engineering practice.

The numerical examples are given. Moreover, the practical application research is implemented for servo control. According to the results, the validity is confirmed. The proposed methods have smaller phase lags and also satisfactory noise tolerance. By using the new approaches, the accuracy of the servo system is effectively improved.

In the future work, the adaptive law of the parameter R will be investigated for the proposed methods. Although the experiments in practical servo system show the effectiveness of the proposed structures, the noise in the estimated velocity response is also observable when the speed of the system is slow, however, is small in the state of high speed. Therefore, fixed R is not optimal. Future research will focus on this limitation by adjusting R according to the working condition of the system.

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Appendix A. Recently, the convergence of the tracking differentiator is rigorously proved by B. Guo and Z. Zhao [16]. According to the proof process, the main results are summarized as the following format.

Lemma A.1. *Suppose that the equilibrium point* $\mathbf{0}$ *of the following system is globally asymptotically stable:*

$$\dot{\mathbf{z}}(t) = F[\mathbf{z}(t)], \quad \mathbf{z}(0) = \mathbf{z}_0, \tag{29}$$

where

$\mathbf{z}(t) = [z_1(t), z_2(t), \ldots, z_n(t)]^T$ *is a n-dimension argument vector;*
$F[\mathbf{z}(t)] = [f_1(\mathbf{z}), f_2(\mathbf{z}), \ldots, f_n(\mathbf{z})]^T$ *is a function vector;*
$\mathbf{z}_0$ *is any initial value.*

There exists a smooth, positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ *and a continuous, positive define function* $W : \mathbb{R}^n \to \mathbb{R}$ *such that: (1)* $V(\mathbf{z}) \to +\infty$ *as* $|\mathbf{z}| \to +\infty$*; (2)* $\frac{dV(\mathbf{z})}{dt} = \frac{\partial V}{\partial z_1}\dot{z}_1 + \frac{\partial V}{\partial z_2}\dot{z}_2 + \ldots + \frac{\partial V}{\partial z_n}\dot{z}_n \le -W(\mathbf{z})$*; (3)* $\{\mathbf{z} \in \mathbb{R}^n | V(\mathbf{z}) \le d\}$ *is a bounded closed set of* $\mathbb{R}^n$ *for any given* $d > 0$.

If there is a perturbed system of (29)

$$\dot{\mathbf{y}}_R(t) = F[\mathbf{y}_R(t)] + \mathbf{g}_R(t), \quad \mathbf{y}_R(0) = \mathbf{y}_{R0}, \tag{30}$$

where

$\mathbf{y}_R(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T$ *is a n-dimension vector;*
$\mathbf{g}_R(t) = \frac{1}{R}\mathbf{a}(t)$ *is a perturbation term,* $\mathbf{a}(t) = [a_1(t), a_2(t), \ldots, a_n(t)]^T$, $\sup_{t\in[0,+\infty)} |a_i(t)|$ *is bounded constant* A_i *for* $i = 1, 2, \ldots, n$*;*
$\mathbf{y}_R(0)$ *is non-increasing with* R*,*

then there exists an R_* *such that* $|\mathbf{y}_R(Rt)| < \varepsilon$ *for any* $\varepsilon > 0$ *and* $a > 0$ *when* $R > R_*$ *and* $t \in [a, +\infty)$.

Remark A.1. *This lemma is summarized from the proof process of the Theorem 2.1 in* [16]*. It was proved as the following line.*

Firstly, due to the existence of the functions $V(\mathbf{z})$ *and* $W(\mathbf{z})$*, there exist class* $\mathcal{K}$ *functions* $K_i : [0, +\infty) \to [0, +\infty)$*,* $i = 1, 2, 3, 4$ *such that*

$$K_1(|\mathbf{z}|) \le V(\mathbf{z}) \le K_2(|\mathbf{z}|), \quad \lim_{r\to+\infty} K_i(r) = +\infty, \quad i = 1, 2,$$

$$K_3(|\mathbf{z}|) \le W(\mathbf{z}) \le K_4(|\mathbf{z}|).$$

Use $\mathbf{y}_R(t;0,\mathbf{y}_{R0})$ *to express the solution of (30), which means that* $\mathbf{y}$ *changes as* t *and* R *with the situation of equilibrium point* $\mathbf{0}$ *and initial value* $\mathbf{y}_{R0}$.

Secondly, according to the monotonicity of the class $\mathcal{K}$ *functions and using the method of proof by contradiction, it is proved that for each* $\mathbf{y}_{R0} \in \mathbb{R}^n$, *there exists an* $R_1 > 1$ *such that when* $R > R_1$,

$$\{\mathbf{y}_R(t;0,\mathbf{y}_{R0})|t \in [0,+\infty)\} \subset \{\mathbf{y}|V(\mathbf{y}) \leq c\}, \quad c = \max\{K_2(|\mathbf{y}_{10}|),1\} > 0.$$

Thirdly, according to the result of the second step and also using the method of proof by contradiction, it is proved that an $R_2 \geq R_1$ *exists such that for* $R > R_2$, *there is a* $T_R \in \left[0, \frac{2c}{K_3 K_2^{-1}(\delta)}\right]$ *such that* $|\mathbf{y}_R(T_R;0,\mathbf{y}_{R0})| < \delta$. *Here,* $\delta > 0$ *is any constant.*

Finally, it is proved that for each $R > R_2$, *if there exists a* $t_0^R \in [0,+\infty)$ *such that*

$$\mathbf{y}_R(t_0^R;0,\mathbf{y}_{R0}) \in \{\mathbf{y}|\ |\mathbf{y}| \leq \delta\},$$

then, for any $\varepsilon > 0$ *there is*

$$\{\mathbf{y}_R(t;0,\mathbf{y}_{R0})|t \in (T_0^R,+\infty)\} \subset \{\mathbf{y}|\ |\mathbf{y}| \leq \varepsilon\}.$$

Combining the third step and the fourth step, there exists an R_* *such that* $|\mathbf{y}_R(Rt)| < \varepsilon$ *for any* $\varepsilon > 0$ *when* $R > R_*$ *and* $a > 0$, $t \in [a,+\infty)$.

Remark A.2. *In fact, the validity of this lemma is guaranteed by the existence of the Lyapunov function of the original system (29). Moreover, the supremum of the perturbation is bounded in the perturbed system (30), and the term of perturbation decreases as* R *increases.*