Switching from subluminal to superluminal light propagation via a coherent pump field in a four-level atomic system

Shang-qi Kuang,[1,2,3] Ren-gang Wan,[1,2] Jun Kou,[1,2] Yun Jiang,[1,2] and Jin-yue Gao[1,2,*]

[1]*College of Physics, Jilin University, Changchun 130023, China*
[2]*Key Lab of Coherent Light, Atomic and Molecular Spectroscopy, Educational Ministry of China, Changchun 130021, China*
[3]*Changchun Institute of Optics, Fine Mechanics and Physics, Chinese Academy of Sciences, Changchun 130033, China*
**Corresponding author: jygao@mail.jlu.edu.cn*

Received June 1, 2009; revised September 24, 2009; accepted October 12, 2009; posted October 15, 2009 (Doc. ID 111931); published November 5, 2009

We theoretically investigate the influence of a coherent pump field on the propagation of a weak light pulse of a probe field in a four-level atomic system. Due to the modulation of the pump field, the light pulse can be manipulated from subluminal to superluminal with negligible distortion. This scheme can be realized in both the ultracold and Doppler-broadened atomic systems. We also demonstrate that the spectral linewidth with an anomalous dispersion is reduced by thermal averaging; therefore, one can obtain a larger negative group refractive index in room-temperature vapor than the largest value achieved in ultracold atomic gas. © 2009 Optical Society of America

OCIS codes: 270.5530, 270.1670, 020.1670.

1. INTRODUCTION

The purpose of research on the manipulation of light group velocity from subluminal to superluminal is not only to study a novel state of matter [1], but also to realize potential applications in the field of optical memory, optical computing, and optical communication [2,3]. There have been many attempts to obtain both ultraslow and fast lights in a single system of different media [1–6], and the related studies have became a research focus of interest. The phenomena of slow and superluminal light propagation have been observed experimentally in a solid with saturated and reverse absorption [4], and the corresponding theoretical explanation was also supplied [5]. In the electromagnetically induced transparency (EIT) medium of cold atoms, the subluminal or superluminal light propagation can be selectively obtained by controlling the intensity of a pump field [1] or switching the circular polarization of the coupling field [7]. In the Λ-type and V-type atomic systems, group light speed can be modulated freely by the intensity and phase of a microwave [8–10]. As an optional method, the group velocity can be varied by the magnitude of a magnetic field in a four-level atomic system [11]. The squeezed field [12] or incoherent pumping field [13,14] can also modulate the group velocity of light pulses arbitrarily in the atomic systems with the effect of spontaneously generated coherence (SGC). However, there are some fundamental limits in the above proposals in which either a coupling of the dipole transition forbidden levels or some special level of structure is required [8–10,12–14]. Furthermore, the light pulses with a superluminal propagation induced by the effect of electromagnetically induced absorption (EIA) have pronounced distortions [11].

In this paper, we theoretically investigate the effect of a coherent pump field on the propagation of a weak probe pulse in a four-level atomic system. Such a system has been used to observe gain of the probe field in cold and hot atomic systems [15–17], while here we found the coherent pump field can be used as a knob to switch the propagation of the probe field from subluminal to superluminal in both cold and hot atoms. To illustrate the advantages of our scheme, we note that the superluminal light propagation in our model is induced by a gain doublet of the probe field; therefore, the light pulses can propagate without distortions, and this advantage increases the practical applicability of our proposal. Furthermore, because atomic motion often masks the coherent effect, some approaches mentioned above can only be realized in the absence of the Doppler broadening effect [1,6–10,12–14]. Although it was demonstrated that pulse propagation could be modulated from subluminal to superluminal in hot Cs vapor [18] and hot Rb vapor, respectively, with a buffer gas [19], the fast light experiences a strong absorption therein. In contrast to these schemes, which can only be realized in a system without the Doppler effect, our scheme is naturally Doppler free, though the configuration of laser fields is not a Doppler-free geometry. In addition, we have found an interesting result in that the spectral linewidth with an anomalous dispersion is reduced by thermal averaging, and one can obtain a larger negative group refractive index in room-temperature vapor than the largest value achieved in ultracold atomic gas.

0740-3224/09/122256-5/$15.00

2. ATOMIC MODEL AND EQUATIONS

The atomic system under consideration can be described by a four-level atomic configuration as shown in Fig. 1(a). We consider the propagation of a light pulse whose central frequency ω_p is resonance with the transition $|3\rangle - |1\rangle$, and the coupling field with a frequency ω_c and coherent pump field with a frequency ω_s interact with the transitions $|3\rangle - |2\rangle$ and $|4\rangle - |1\rangle$, respectively. Here $2\gamma_i$ ($i = 1,2,3,4$) is the spontaneous emission of the corresponding transition, and the transition $|2\rangle - |1\rangle$ is electric dipole forbidden.

In the framework of the semiclassical theory using the dipole approximation and the rotating wave approximation, the Hamiltonian H_I of the system in the interaction picture is

$$H_I = \hbar(\Delta_c - \Delta_p)|2\rangle\langle 2| - \hbar\Delta_p|3\rangle\langle 3| - \hbar\Delta_s|4\rangle\langle 4| - \hbar[g|3\rangle\langle 1| + \Omega_c|3\rangle\langle 2| + \Omega_s|4\rangle\langle 1| + H.c.], \tag{1}$$

where $\Delta_p = \omega_p - \omega_{31}$, $\Delta_c = \omega_c - \omega_{32}$, and $\Delta_s = \omega_s - \omega_{41}$ are the detunings of the three laser fields. The Rabi frequencies of the probe, coupling, and pump fields are $g = \mu_{31}E_p/2\hbar$, $\Omega_c = \mu_{32}E_c/2\hbar$, and $\Omega_s = \mu_{41}E_s/2\hbar$, respectively. For simplicity, we take these Rabi frequencies as real. Including relaxation terms for the closed system, the equations of motion for the density matrix of the four-level system are

$$\dot\rho_{22} = 2\gamma_2\rho_{33} + 2\gamma_4\rho_{44} + i\Omega_c(\rho_{32} - \rho_{23}),$$

$$\dot\rho_{33} = -2(\gamma_1 + \gamma_2)\rho_{33} + ig(\rho_{13} - \rho_{31}) + i\Omega_c(\rho_{23} - \rho_{32}),$$

$$\dot\rho_{44} = -2(\gamma_3 + \gamma_4)\rho_{44} + i\Omega_s(\rho_{14} - \rho_{41}),$$

$$\dot\rho_{31} = \Gamma_{31}\rho_{31} + ig(\rho_{11} - \rho_{33}) + i\Omega_c\rho_{21} - i\Omega_s\rho_{34},$$

$$\dot\rho_{32} = (i\Delta_c - \gamma_1 - \gamma_2)\rho_{32} + ig\rho_{12} + i\Omega_c(\rho_{22} - \rho_{33}),$$

$$\dot\rho_{21} = \Gamma_{21}\rho_{21} + i\Omega_c\rho_{31} - ig\rho_{23} - i\Omega_s\rho_{24},$$

$$\dot\rho_{14} = -(i\Delta_s + \gamma_3 + \gamma_4)\rho_{14} - i\Omega_s(\rho_{11} - \rho_{44}) + ig\rho_{34},$$

$$\dot\rho_{24} = \Gamma_{24}\rho_{24} - i\Omega_s\rho_{21} + i\Omega_c\rho_{34},$$

$$\dot\rho_{34} = \Gamma_{34}\rho_{34} - i\Omega_s\rho_{31} + i\Omega_c\rho_{24} + ig\rho_{14},$$

$$1 = \rho_{11} + \rho_{22} + \rho_{33} + \rho_{44}, \tag{2}$$

where $\Gamma_{31} = i\Delta_p - \gamma_1 - \gamma_2$, $\Gamma_{21} = i(\Delta_p - \Delta_c) - \gamma_{21}$, $\Gamma_{24} = i(\Delta_p - \Delta_s - \Delta_c) - (\gamma_3 + \gamma_4)$, and $\Gamma_{34} = i(\Delta_p - \Delta_s) - (\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)$. Because the probe filed is weak ($g \ll \Omega_c, \Omega_s, 2\gamma_1$), we solve the above density matrix and derive the first-order steady-state solution of the element $\rho_{31}^{(1)}$ as

$$\frac{\rho_{31}^{(1)}}{g} = \frac{i(\rho_{33}^{(0)} - \rho_{11}^{(0)})\mathcal{M} + \Omega_s\mathcal{R}\rho_{14}^{(0)} + \Omega_c\mathcal{F}\rho_{23}^{(0)}}{\Gamma_{31}\mathcal{M} + \Omega_c^2\mathcal{F} + \Omega_s^2\mathcal{R}}, \tag{3}$$

where $\mathcal{M} = \Gamma_{21}\Gamma_{34}\Gamma_{24} + \Gamma_{34}\Omega_s^2 + \Gamma_{21}\Omega_c^2$, $\mathcal{R} = \Gamma_{21}\Gamma_{24} + \Omega_s^2 - \Omega_c^2$, and $\mathcal{F} = \Gamma_{24}\Gamma_{34} + \Omega_c^2 - \Omega_s^2$. Here γ_{21} is the dephasing rate between the states $|2\rangle$ and $|1\rangle$. For simplicity, we assume $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma/2$, and the zero-order results of the terms in Eq. (3) are given as

$$\rho_{33}^{(0)} = \frac{\Omega_s^2\Omega_c^2}{\Omega_s^2(2\Omega_c^2 + \gamma^2 + \Delta_c^2) + \Omega_c^2(2\Omega_s^2 + \gamma^2 + \Delta_s^2)},$$

$$\rho_{11}^{(0)} = \frac{\Omega_s^2 + \gamma^2 + \Delta_s^2}{\Omega_s^2}\rho_{33}^{(0)},$$

$$\rho_{23}^{(0)} = -\frac{\Delta_c + i\gamma}{\Omega_c}\rho_{33}^{(0)},$$

$$\rho_{14}^{(0)} = -\frac{\Delta_s + i\gamma}{\Omega_s}\rho_{33}^{(0)}. \tag{4}$$

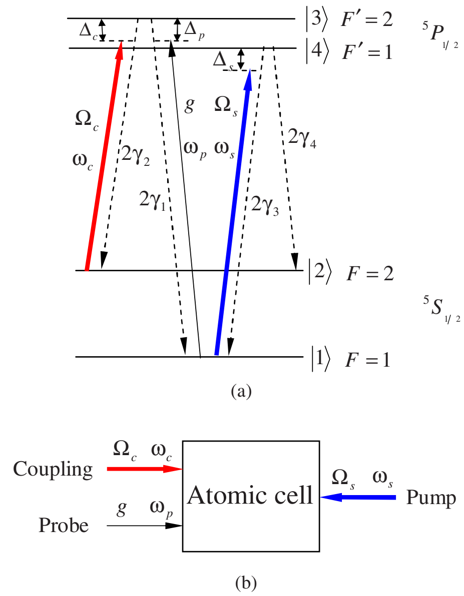

Fig. 1. (Color online) (a) Schematic diagram of a four-level atomic system. (b) Block diagram where the coupling (ω_c) and probe (ω_p) fields are copropagating and the coherent pump field (ω_s) is counterpropagating inside the medium.

Considering the Doppler frequency shifts induced by the atoms with velocity v, we substitute Δ_p, Δ_s, and Δ_c with $\Delta_p + \omega_p v$, $\Delta_s - \omega_s v$, and $\Delta_c + \omega_c v$, which correspond to the configuration of laser fields as shown in Fig. 1(b). From Eq. (3) we obtain the linear susceptibility $\chi(\omega_p)$, which is averaged over the Doppler distribution of atomic velocities as

$$\chi = 3\pi\gamma\mathcal{N}\int_{-\infty}^{+\infty}\frac{\rho_{31}^{(1)}}{g}f(v)dv. \tag{5}$$

Here $\mathcal{N} = N(\lambda_p/2\pi)^3$ is the scaled average atomic density, N presents the atomic density, and λ_p is the wavelength of the probe field. $f(v) = \exp(-v^2/v_p^2)/v_p\sqrt{\pi}$ is the Maxwellian distribution, and $v_p = \sqrt{2kT/M}$ represents the most probable atomic velocity. We introduce the group refractive index $n_g = c/v_g$, where c is the speed of light in vacuum, and the group velocity v_g is given by [20]

$$v_g = \frac{c}{1 + \frac{1}{2}\mathrm{Re}(\chi) + \frac{\omega_p}{2}\frac{\partial \mathrm{Re}(\chi)}{\partial \omega_p}}. \quad (6)$$

3. RESULTS AND DISCUSSION

For calculations, we adopt the transition $^5S_{1/2}$ to $^5P_{1/2}$ of ^{87}Rb as shown in Fig. 1(a). First, we display the real and imaginary parts of susceptibility $\chi(\omega_p)$ of the probe field in hot atoms in Fig. 2, which correspond to the dispersion and absorption of the probe field, respectively. It is shown that one can obtain a sub-Doppler spectral resolution of the probe field in the Doppler-broadened system, though the configuration of the laser fields is not a three-photon Doppler-free geometry. When the intensity of the pump field becomes larger than the coupling field, the probe field is amplified at the frequencies around the resonance point, and the dispersion of the probe field changes from normal to anomalous.

In order to interpret these results, we supply an analysis in the dressed-state picture for the system. The interaction Hamiltonian for the atoms with velocity v, which are driven by the coupling and pump fields, can be written as [21]

$$H_I = -\hbar\begin{pmatrix} 0 & 0 & 0 & \Omega_s \\ 0 & -\Delta_c - kv & \Omega_c & 0 \\ 0 & \Omega_c & 0 & 0 \\ \Omega_s & 0 & 0 & \Delta_s - kv \end{pmatrix}. \quad (7)$$

Here we set $k_c = k_s = k_p = k$ as an approximation and consider the resonance condition that $\Delta_c = \Delta_s = 0$. From the eigenvalues of the four dressed states, we obtain the splitting of the dressed-state transitions, which the probe field couples as

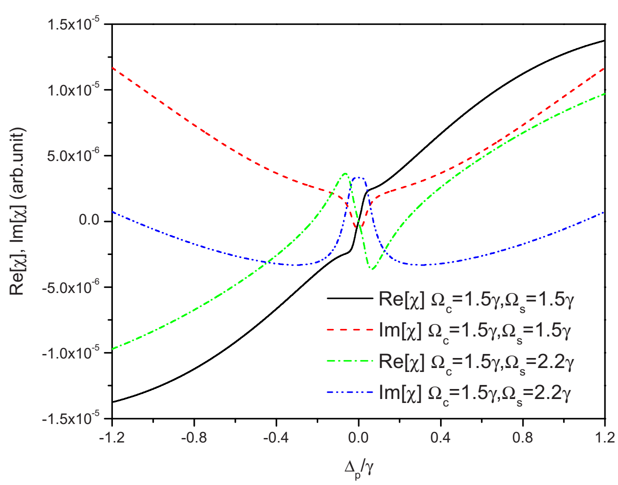

Fig. 2. (Color online) Real and imaginary parts of susceptibility χ versus probe frequency ω_p in the presence of the coupling field Ω_c and the pump field Ω_s in hot atoms. The common parameters of the above curves are chosen as atomic density $N=2\times 10^{11}$ cm^{-3}, $2\gamma/2\pi = 5.746$ MHz, and $\gamma_{21} = 0.001\gamma$. $\Delta_c = \Delta_s = 0$, and we assume the most probable velocity as $v_p = 250$ m/s.

$$\alpha_1 = -\hbar\left(\sqrt{(kv/2)^2 + \Omega_c^2} + \sqrt{(kv/2)^2 + \Omega_s^2}\right),$$

$$\alpha_2 = -\hbar\left(\sqrt{(kv/2)^2 + \Omega_c^2} - \sqrt{(kv/2)^2 + \Omega_s^2}\right),$$

$$\alpha_3 = \hbar\left(\sqrt{(kv/2)^2 + \Omega_c^2} - \sqrt{(kv/2)^2 + \Omega_s^2}\right),$$

$$\alpha_4 = \hbar\left(\sqrt{(kv/2)^2 + \Omega_c^2} + \sqrt{(kv/2)^2 + \Omega_s^2}\right). \quad (8)$$

For stationary atoms, due to the population distribution of the excited state $\rho_{33}^{(0)}$ and the constructive interference effect $\rho_{14}^{(0)}$ induced by the pump field Ω_s as shown in Eq. (3), there are amplifications of the probe field at two transitions that correspond to the two dressed states α_2 and α_3 in Eq. (8). Because the population of level $|1\rangle$ is always larger than that of level $|3\rangle$ ($\rho_{11}^{(0)} > \rho_{33}^{(0)}$) as shown in Eq. (4), the probe field is amplified without the population inversion [15]. When the intensity of the pump field is small, $\Omega_s \leq \Omega_c$, the two dressed states become degenerate, and then the probe field has one transition with a normal dispersion. As the pump field intensity increases, $\Omega_s > \Omega_c$, the two dressed states become distinct, and then the probe field has two gain lines with an anomalous dispersion. For atoms with velocity v, the two dressed states α_2 and α_3 do not shift much with kv due to the partial cancellation of this term in Eq. (8). As a result, we still can obtain the two gain lines of the probe field after considering the effect of the Doppler broadening, as shown in Fig. 2. Therefore, one can switch the dispersion of the system from normal to anomalous by increasing the intensity of the coherent pump field in both cold and hot atoms.

Second, we supply the calculated group refractive index n_g at the resonance frequency of the probe field in cold and hot atoms as a function of Ω_s with different intensities of the coupling field in Fig. 3(a). It is found that the group refractive index can be modulated from positive to negative by the pump field, and the largest negative group refractive index in cold atoms is smaller than that achieved in hot atoms. This result is that there is an optimal width of the probe absorption with anomalous dispersion in cold atoms. While due to the contributions of atoms with different velocities, as shown in Fig. 3(b), the region with an anomalous dispersion can be narrower than that in stationary atoms. As a result, one can have a larger group refractive index n_g due to the Kramer–Kronig relation [22] in hot atoms. This interesting result due to the Doppler broadening effect is similar with the latest experimental report [23]. Here we also investigate the corresponding transmission of the probe field propagating through a medium of length $L = 1$ cm and show the results as an inset in Fig. 3(a). The inset shows that one can obtain superluminal light propagation with a small absorption in both cold and hot atoms, which is very helpful to observe the phenomenon of superluminal light propagation. It is worthwhile to point out that though the transition $|4\rangle - |2\rangle$ is allowed in our system, we ignore the four-wave mixing signal and its effect on the probe field, owing to the limited optical density length and the weak probe field.

In the following, we consider the propagation of a

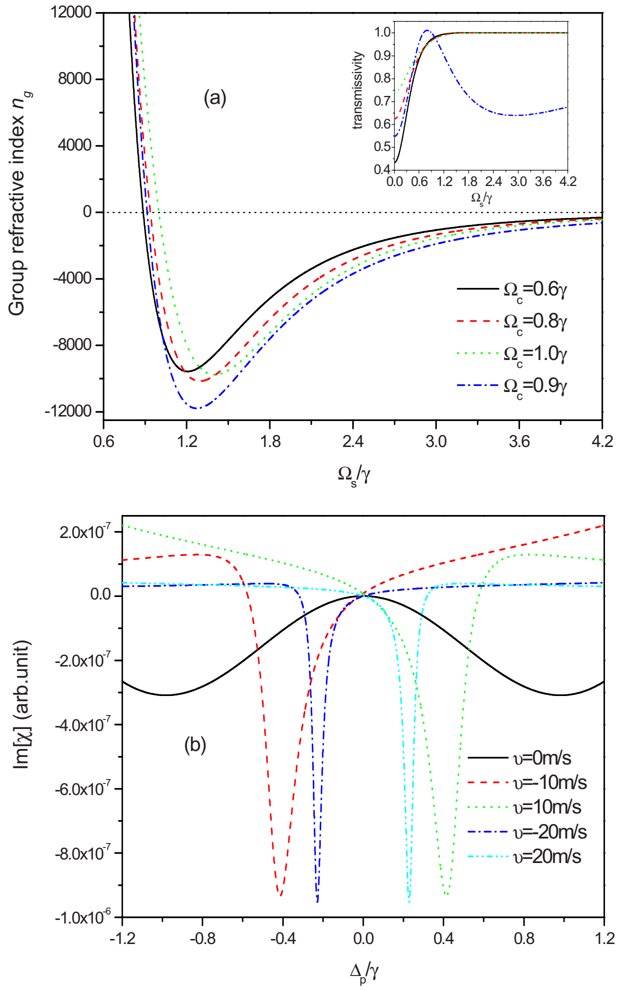

Fig. 3. (Color online) (a) Group refractive index n_g at resonance of the probe field in the presence of the coupling field $\Omega_c = 0.6, 0.8, 1.0\gamma$ in cold atoms and in the presence of the coupling field $\Omega_c = 0.9\gamma$ in hot atoms. The inset shows the corresponding transmission of the probe field that propagates through a medium of length $L = 1$ cm. Other parameters are the same as in Fig. 2. (b) Effect of the velocity on probe absorption in the presence of the coupling and pump fields $\Omega_c = 0.8\gamma$ and $\Omega_s = 1.2\gamma$.

Gaussian pulse through the sample with a length $L = 1$ cm to confirm the above results. We obtain the solution of the field profile for the probe pulse as [24]

$$E(z,t) = \int_{-\infty}^{+\infty} \varepsilon(0,\omega_p) \exp\left[-i\omega_p \left(t - \frac{zn(\omega_p)}{c} \right) \right] d\omega_p, \quad (9)$$

where $\varepsilon(0,\omega_p) = \tau/(2\pi)^{1/2} \exp[-(\omega_p - \omega_{31})^2 \tau^2/2]$ is the Fourier transform of the probe field at the entrance of the cell, z is the propagation distance, and $n(\omega_p) = \sqrt{1 + \chi(\omega_p)}$ is the refractive index. For a pulse with $\tau = 2$ μs, we show the delays due to the medium under different conditions in Fig. 4. Calculating the relative delays between the reference pulse and the output pulses, we obtain the group velocities of the pulses and find that there are very good agreements with the results shown in Fig. 3(a). Here we confirm that the spectral width of the Gaussian pulse is well contained within the region that is between two closely spaced gain lines, so there are no distortions for the propagation of the pulses. Moreover, the transmissions of the Gaussian pulses in Fig. 4 agree with the corresponding results shown in the inset in Fig. 3(a). As a result, it is demonstrated in Fig. 4 that the pump field can be a knob to switch the propagation of a light pulse from subluminal to superluminal, or vice versa, in cold or hot atoms.

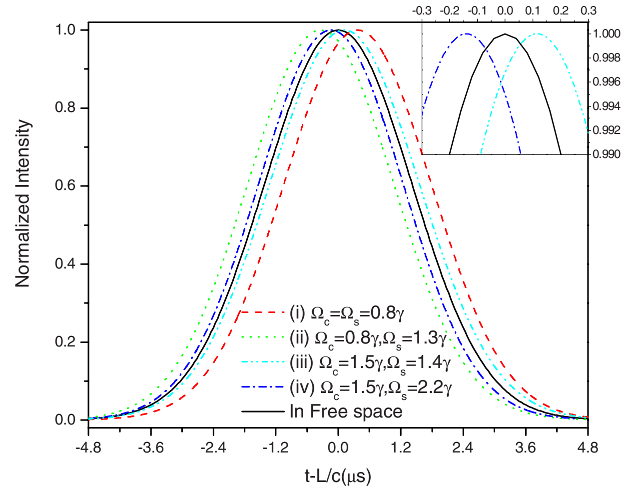

Fig. 4. (Color online) Solid curve represents a Gaussian pulse propagating at speed c through 1 cm of vacuum (reference). Curves (i) and (ii) show the same pulse propagation through cold atoms of length 1 cm with time delays 0.36 μs and -0.34 μs, respectively. Curves (iii) and (iv) show the same pulse propagation through a 1 cm long hot atomic vapor with time delays 0.12 μs and -0.14 μs, respectively. We chose cold and hot atomic systems that have the same atomic density as $N = 2 \times 10^{11}$ cm^{-3}; other common parameters are the same as Fig. 2. The inset shows a magnified part of the same.

4. CONCLUSIONS

In a four-level atomic system, we theoretically present that a weak probe field can be switched from subluminal to superluminal propagation by a coherent pump field. Our proposal can be realized in atomic systems with or without the Doppler effect, and the superluminal light propagation induced by a doublet gain can get no distortion. Owing to the Doppler averaging effect, linewidth with an anomalous dispersion is reduced, and the largest negative group refractive index in ultracold atomic gas is smaller than that one can achieve in hot atomic vapor.

ACKNOWLEDGMENTS

This work is supported by the National Science Foundation of China (NSFC) grants 10334010 and 10774059, the doctoral program foundation of the Institution of Higher Education of China, and the National Basic Research Program grant 2006CB921103.

REFERENCES

1. H. Kang, G. Hernandez, and Y. F. Zhu, “Superluminal and slow light propagation in cold atoms,” Phys. Rev. A **70**, 011801 (2004).
2. H. Y. Tseng, J. Huang, and A. Adibi, “Expansion of the relative time delay by switching between slow and fast light using coherent population oscillation with semiconductors,” Appl. Phys. B **85**, 493–501 (2006).
3. D. Dahan and G. Eisenstein, “Tunable all optical via slow

and fast light propagation in a Raman assisted fiber optical parametric amplifier: a route to all optical buffering," Opt. Express **13**, 6234–6249 (2005).
4. M. S. Bigelow, N. N. Lepeshkin, and R. W. Boyd, "Superluminal and slow light propagation in a room-temperature solid," Science **301**, 200–202 (2003).
5. G. S. Agarwal and T. N. Dey, "Sub- and superluminal propagation of intense pulses in media with saturated and reverse absorption," Phys. Rev. Lett. **92**, 203901 (2004).
6. Z. Haghshenasfard, M. H. Naderi, and M. Soltanolkotabi, "Subluminal to superluminal propagation of an optical pulse in an *f*-deformed Bose–Einstein condensate," J. Phys. B **41**, 165501 (2008).
7. J. Zhang, G. Hernandez, and Y. Zhu, "Copropagating superluminal and slow light manifested by electromagnetically assisted nonlinear optical processes," Opt. Lett. **31**, 2598–2600 (2006).
8. G. S. Agarwal, T. N. Dey, and S. Menon, "Knob for changing light propagation from subluminal to superluminal," Phys. Rev. A **64**, 053809 (2001).
9. H. Sun, H. Guo, Y. Bai, D. Han, S. Fan, and Xu Chen, "Light propagation from subluminal to superluminal in a three-level Λ-type system," Phys. Lett. A **335**, 68–75 (2005).
10. D. Bortman-Arbiv, A. D. Wilson-Gordon, and H. Friedmann, "Phase control of group velocity: from subluminal to superluminal light propagation," Phys. Rev. A **63**, 043818 (2001).
11. Y. M. Golubev, T. Y. Golubeva, Y. V. Rostovtsev, M. O. Scully, "Control of group velocity of light via magnetic field," Opt. Commun. **278**, 350–362 (2007).
12. F. Carreño, O. G. Calderón, M. A. Antón, and I. Gonzalo, "Superluminal and slow light in Λ-type three-level atoms via squeezed vacuum and spontaneously generated coherence," Phys. Rev. A **71**, 063805 (2005).
13. M. Mahmoudi, M. Sahrai, and H. Tajalli, "The effects of incoherent pumping field on the phase control of group velocity," J. Phys. B **39**, 1825–1835 (2006).
14. M. Mahmoudi, M. Sahrai, and H. Tajalli, "Subluminal and superluminal light propagation via interference of incoherent pump fields," Phys. Lett. A **357**, 66–71 (2006).
15. H. Kang, L. L. Wen, and Y. F. Zhu, "Normal or anomalous dispersion and gain in a resonant coherent medium," Phys. Rev. A **68**, 063806 (2003).
16. L. B. Kong, X. H. Tu, J. Wang, Y. F. Zhu, and M. S. Zhan, "Sub-Doppler spectral resolution in a resonantly driven four-level coherent medium," Opt. Commun. **269**, 362–369 (2007).
17. W. H. Xu and J. Y. Gao, "Influence of Doppler-broadening on absorption-dispersion properties in a resonant coherent medium," Chin. Phys. **14**, 2496–2502 (2005).
18. K. Kim, H. S. Moon, C. Lee, S. K. Kim, and J. B. Kim, "Observation of arbitrary group velocities of light from superluminal to subluminal on a signal atomic transition line," Phys. Rev. A **68**, 013810 (2003).
19. E. E. Mikhailov, V. A. Sautenkov, Y. V. Rostovtsev, and G. R. Welch, "Absorption resonance and large negative delay in rubidium vapor with a buffer gas," J. Opt. Soc. Am. B **21**, 425–428 (2004).
20. S. E. Harris, J. E. Field, and A. Kasapi, "Dispersive properties of electromagnetically induced transparency," Phys. Rev. A **46**, R29–R32 (1992).
21. C. Y. Ye, A. S. Zibrov, Yu. V. Rostovtsev, and M. O. Scully, "Unexpected Doppler-free resonance in generalized double dark states," Phys. Rev. A **65**, 043805 (2002).
22. J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, 1975).
23. S. M. Iftiquar, G. R. Karve, and V. Natarajan, "Subnatural linewidth for probe absorption in an electromagnetically induced transparency medium due to Doppler averaging," Phys. Rev. A **77**, 063807 (2008).
24. C. G. B. Garrett and D. E. Mocumber, "Propagation of a Gaussian light pulse through an anomalous dispersion medium," Phys. Rev. A **1**, 305–313 (1970).