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Fast communication

Reduced biquaternion canonical transform, convolution and correlation

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ABSTRACT

The reduced biquaternion canonical transform (RBCT) is defined in this paper, which is the generalization of reduced biquaternion Fourier transform (RBFT). The Parseval's theorem related to RBCT is investigated. The concepts of reduced biquaternion canonical convolution (RBCCV) and reduced biquaternion canonical correlation (RBCCR) are defined, then the convolution and correlation theorem of RBCT are developed in this paper. All these theorems can also be seen as the generalizations of the corresponding theorem related to RBFT. Finally, the discrete form and fast algorithm of RBCT are presented, and the computation complexity is similar to FFT.

1. Introduction

Recently, many signal processing tools using the quaternions have been proposed, including quaternion Fourier transform (QFT) [1–4], quaternion wavelet transform (QWT) [5–7] and fractional quaternion Fourier transform (FrQFT) [8]. However, due to the noncommutative property of the quaternion multiplication, some important theorems cannot be generalized to quaternion signal processing tools. For example, the Parseval's theorem does not hold for FrQFT; the convolution of two quaternion signal $f(x,y)$ and $g(x,y)$ cannot be calculated by the product of their QFT [3]. In addition, the computation of QFT and FrQFT are somewhat complex. In order to overcome these drawbacks, we combine the reduced biquaterion algebra and the linear canonical transform (LCT) and propose the reduced biquaterion canonical transform (RBCT).

Linear canonical transform (LCT) [10–14] is an important tool in signal processing. Many transforms such as Fourier transform, fractional Fourier transform and the Fresnel transform are special cases of the LCT. During the last decade, there were many achievements associated with the LCT [15–21]. However, the LCT deals with real scalar signal or complex signal (analytical signal [19]), none of them processes reduced biquaternion signals. As the generalization of complex signal, reduced biquaternion signal, has one real part and three imaginary parts. Based on reduced biquaternion (RB) algebra system, we define the reduced biquaternion canonical transform, which can process reduced biquaternion signals.

In Section 2, we first give a brief introduction about quaternion and reduced biquaternion. The reduced biquaternion canonical transform is defined, which can process not only real scalar or complex signal but also reduced biquaternion signal. The Parseval's equality of RBCT is derived in Section 3. Moreover, in Sections 4 and 5, we generalize the definitions of convolution and correlation, propose the reduced biquaternion canonical convolution (RBCCV) and reduced biquaternion canonical correlation

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(RBCCR). The convolution and correlation theorems of RBCT are derived. The theorems show that the RBCCV or RBCCR can be calculated in the RBCT domain. Section 6 developed the discrete form and fast algorithm of RBCT, simulations are also implemented. Finally, conclusions are made in Section 7.

2. Preliminaries

2.1. The quaternion signal and reduced biquaternion signal

The quaternion, which is a type of hypercomplex number, was formally introduced by Hamilton in 1843. It is a generalization of complex number. We know that a complex number has two components: the real part and imaginary part. However, the quaternion has four parts, i.e., one real part and three imaginary parts. For a quaternion q, which can be written in a rectangular form as follows: $q=q_r+q_i i+q_j j+q_k k$, where $q_r,q_i,q_j,q_k \in \mathbb{R}$ and i,j,k are complex operators obeying the following rules:

$$i^2=j^2=k^2=-1, \quad ij=-ji=k,\ jk=-kj=i,\ ki=-ik=j \tag{1}$$

Correspondingly, the complex signal can be generalized to quaternion signal, i.e., $f(x,y)=f_r(x,y)+if_i(x,y)+jf_j(x,y)+kf_k(x,y)$, where $f_r(x,y),f_i(x,y),f_j(x,y),f_k(x,y) \in \mathbb{R}^2$. Some signal processing tools can process quaternion signals, including quaternion Fourier transform, quaternion wavelet transform, fractional quaternion Fourier transform and quaternion Fourier–Mellin moment [9].

However, the multiplication rule of quaternions is not commutative. T.A.Ell defined the double-complex algebra, which is similar to quaternions but with commutative multiplication [1]. In Ref. [22], reduced biquaternions (RBs) was proposed, and commutative multiplication was defined for it.

The description of RBs is $q=q_r+q_i i+q_j j+q_k k$, where $q_r,q_i,q_j,q_k \in \mathbb{R}$, the imaginary operators obey the following multiplicative rules:

$$i^2=k^2=-1, \quad j^2=1,\ ij=ji=k,\ jk=kj=i,\ ki=ik=-j \tag{2}$$

RBs can also be represented as follows: $q=q_1+q_2 j$, where $q_1=q_r+q_i i$, $q_2=q_j+q_k i$.

Another representation of RBs is e_1-e_2 form [23], i.e., $q=q_1+q_2 j \equiv q_{1+2}e_1+q_{1-2}e_2$, where $q_{1+2}=q_1+q_2$, $q_{1-2}=q_1-q_2$, $e_1=(1+j)/2$, $e_2=(1-j)/2$. We will use e_1-e_2 form of RB signals to develop the fast algorithm of RBCT in Section 6.

In Ref. [24], the norm and conjugate of RBs were defined. The norm of a reduced biquaternion $q=q_r+q_i i+q_j j+q_k k$ is

$$\|q\|=[(q_r^2+q_i^2+q_j^2+q_k^2)^2-4(q_r q_j+q_i q_k)^2]^{\frac{1}{4}} \tag{3}$$

the conjugate of q is

$$\overline{q}=\|q\|^2 q^{-1}=\|q\|^2/q \tag{4}$$

The two-dimensional reduced biquaternion signal $f(x,y)$ is defined as follows: $f(x,y)=f_r(x,y)+if_i(x,y)+jf_j(x,y)+kf_k(x,y)$. For any two reduced biquaternion signals $f(x,y)$ and $g(x,y)$, $f(x,y)=f_1(x,y)+f_2(x,y)j$, $g(x,y)=g_1(x,y)+g_2(x,y)j$, where $f_1(x,y)=f_r(x,y)+if_i(x,y)$, $f_2(x,y)=f_j(x,y)+if_k(x,y)$, $g_1(x,y)=g_r(x,y)+ig_i(x,y)$, $g_2(x,y)=g_j(x,y)+ig_k(x,y)$. According to the multiplication rule of reduced biquaternion, we have

$$f(x,y)g(x,y)=g(x,y)f(x,y)=[f_1(x,y)g_1(x,y)+f_2(x,y)g_2(x,y)] + j[f_1(x,y)g_2(x,y)+f_2(x,y)g_1(x,y)] \tag{5}$$

If $f(x,y)$ and $g(x,y)$ are quaternion signals, Eq. (5) will not holds, i.e., $f(x,y)g(x,y)\neq g(x,y)f(x,y)$.

2.2. Definition of the RBCT

Definition 1. Let $f(x,y)$ be a reduced biquaternion signal with condition $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\|f(x,y)\|^2\,dx\,dy<\infty$, then the RBCT of $f(x,y)$ with parameters A_1 and A_2 is defined as

$$F_{i,k}^{A_1,A_2}(u,v)\triangleq L_{i,k}^{A_1,A_2}[f(x,y)](u,v) = \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)K_i^{A_1}(x,u)K_k^{A_2}(y,v)\,dx\,dy, & b_1b_2\neq 0 \\ \sqrt{d_1d_2}f(d_1u,d_2v)e^{i(c_1d_1/2)u^2}e^{k(c_2d_2/2)v^2}, & b_1b_2=0 \end{cases} \tag{6}$$

where

$$K_i^{A_1}(x,u)=\sqrt{\frac{1}{i2\pi b_1}}e^{i((d_1/2b_1)u^2+(a_1/2b_1)x^2-(1/b_1)ux)} \tag{7}$$

$$K_k^{A_2}(y,v)=\sqrt{\frac{1}{k2\pi b_2}}e^{k((d_2/2b_2)v^2+(a_2/2b_2)y^2-(1/b_2)vy)} \tag{8}$$

$A_1=(a_1,b_1,c_1,d_1)$, $A_2=(a_2,b_2,c_2,d_2)$, $a_s,b_s,c_s,d_s \in \mathbb{R}, s=1,2$. $a_1d_1-b_1c_1=1$, $a_2d_2-b_2c_2=1$.

If $A_1=A_2=(0,1,-1,0)$, the RBCT will be reduced to RBFT [24,25]. If $A_1=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)$, $A_2=(\cos\beta,\sin\beta,-\sin\beta,\cos\beta)$, we can define the fractional reduced biquaternion Fourier transform (FrRBFT). In Ref. [8], the fractional quaternion Fourier transform was defined. Because the definition of FrQFT is based on quaternion algebra, so the commutative property of multiplication does not hold. This drawback leads to that the FrQFT does not satisfy parseval's principle any longer and the computation of the FrQFT is more complex. As a special case of RBCT, the definition of FrRBFT is based on commutative quaternion algebra. So the FrRBFT can overcome the drawback mentioned above. If $f(x,y)$ is the real scalar or complex signal, and set the imaginary operator $k=i$ in Eq. (6), then the RBCT will be reduced to traditional two-dimensional LCT. So the RBCT can also be seen as the generalization of LCT [19,26,27].

The inverse transform of the RBCT is also given by a reduced biquaternion canonical transform with parameters: $A_1^{-1}=(d_1,-b_1,-c_1,a_1)$ and $A_2^{-1}=(d_2,-b_2,-c_2,a_2)$, that is

$$f(x,y)=\begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{i,k}^{A_1,A_2}(u,v)K_i^{A_1^{-1}}(u,s)K_k^{A_2^{-1}}(v,t)\,du\,dv, & b_1b_2\neq 0 \\ \sqrt{a_1a_2}f(a_1x,a_2y)e^{-i(a_1c_1/2)x^2}e^{-k(a_2c_2/2)y^2}, & b_1b_2=0 \end{cases} \tag{9}$$

We give a brief proof of Eq. (9) for the condition $b_1 b_2 \neq 0$:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{i,k}^{A_1,A_2}(u,v)K_i^{A_1^{-1}}(u,s)K_k^{A_2^{-1}}(v,t)\,du\,dv$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)K_i^{A_1}(x,u)K_k^{A_2}(y,v)\,dx\,dy\cdot K_i^{A_1^{-1}}(u,s)K_k^{A_2^{-1}}(v,t)\,du\,dv$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\int_{-\infty}^{\infty} K_i^{A_1}(x,u)K_i^{A_1^{-1}}(u,s)\,du\int_{-\infty}^{\infty} K_k^{A_2}(v,y)K_k^{A_2^{-1}}(t,v)\,dv\,dx\,dy$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x-s)\delta(y-t)\,dx\,dy = f(x,y) \tag{10}$$

For the condition $b_1 b_2 = 0$ in Eq. (6), the RBCT of a reduced biquaternion signal is essentially a chip multiplication. We only discuss the condition $b_1 b_2 \neq 0$ in this work.

3. Parseval's theorem

Theorem 1. *For any two reduced biquaternion signals f(x,y) and g(x,y) with conditions*: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\|f(x,y)\|^2\,dx\,dy<\infty$, *and* $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\|g(x,y)\|^2\,dx\,dy\ <\infty$, *then the following equation holds*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\overline{g(x,y)}\,dx\,dy = \left\|\sqrt{\frac{-1}{i2\pi b_1}}\right\|^2\left\|\sqrt{\frac{-1}{k2\pi b_2}}\right\|^2 \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{i,k}^{A_1,A_2}(u,v)\overline{G_{i,k}^{A_1,A_2}(u,v)}\,du\,dv \tag{11}$$

where $F_{i,k}^{A_1,A_2}(u,v)$ *and* $G_{i,k}^{A_1,A_2}(u,v)$ *are the RBCT of f(x,y) and g(x,y), respectively*.

Proof.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\overline{g(x,y)}\,dx\,dy$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{i,k}^{A_1,A_2}(u,v)K_i^{A_1^{-1}}(u,x)K_k^{A_2^{-1}}(v,y)\,du\,dv\ \overline{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_{i,k}^{A_1,A_2}(s,t)K_i^{A_1^{-1}}(s,x)K_k^{A_2^{-1}}(t,y)\,ds\,dt}\,dx\,dy$$
$$=\left\|\sqrt{\frac{-1}{i2\pi b_1}}\right\|^2\left\|\sqrt{\frac{-1}{k2\pi b_2}}\right\|^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{i,k}^{A_1,A_2}(u,v)\,\overline{G_{i,k}^{A_1,A_2}(s,t)}e^{-i(d_1/2b_1)(u^2-s^2)}e^{-k(d_2/2b_2)(v^2-t^2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(1/b_1)(u-s)x}e^{k(1/b_2)(v-t)y}\,dx\,dy\,du\,dv\,ds\,dt$$
$$=\left\|\sqrt{\frac{-1}{i2\pi b_1}}\right\|^2\left\|\sqrt{\frac{-1}{k2\pi b_2}}\right\|^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{i,k}^{A_1,A_2}(u,v)\,\overline{G_{i,k}^{A_1,A_2}(s,t)}e^{-i(d_1/2b_1)(u^2-s^2)}e^{-k(d_2/2b_2)(v^2-t^2)}\delta(u-s)\delta(v-t)\,ds\,dt\,du\,dv$$
$$=\left\|\sqrt{\frac{-1}{i2\pi b_1}}\right\|^2\left\|\sqrt{\frac{-1}{k2\pi b_2}}\right\|^2\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{i,k}^{A_1,A_2}(u,v)\overline{G_{i,k}^{A_1,A_2}(u,v)}\,du\,dv \qquad \square$$

Corollary 1. *If f(x,y)=g(x,y) in Theorem*1, *then we have the Parseval's principle for the RBCT*:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\|f(x,y)\|^2\,dx\,dy = \left\|\sqrt{\frac{-1}{i2\pi b_1}}\right\|^2\left\|\sqrt{\frac{-1}{k2\pi b_2}}\right\|^2\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\|F_{i,k}^{A_1,A_2}(u,v)\|^2\,du\,dv \tag{12}$$

4. Reduced biquaternion canonical convolution theorem

In this section, the convolution theorem of RBCT is derived. First we give the definition of Reduced biquaternion canonical convolution (RBCCV):

Definition 2. Let *f*(*x*,*y*) and *g*(*x*,*y*) are two reduced biquaternion signals, the RBCCV of *f*(*x*,*y*) and *g*(*x*,*y*) is defined as follows:

$$f(x,y)\circledast g(x,y)=\sqrt{\frac{1}{i2\pi b_1}}\sqrt{\frac{1}{k2\pi b_2}}e^{-i(a_1/2b_1)x^2}e^{-k(a_2/2b_2)y^2}\cdot [e^{i(a_1/2b_1)x^2}e^{k(a_2/2b_2)y^2}f(x,y)]*[g(x,y)e^{i(a_1/2b_1)x^2}e^{k(a_2/2b_2)y^2}] \tag{13}$$

where "*" is the traditional convolution operator. The definition of RBCCV can be seen as the generalization of RB convolution in [25]. If $A_1=A_2=(0,1,-1,0)$, the RBCCV will be reduced to RB convolution.

We have the following Reduced biquaternion canonical convolution theorem:

Theorem 2. *Assume f*(*x*,*y*) *and g*(*x*,*y*) *are two RB signals.* $F_{i,k}^{A_1,A_2}(u,v)$ *and* $G_{i,k}^{A_1,A_2}(u,v)$ *are the RBCT of f*(*x*,*y*) *and g*(*x*,*y*), *respectively*, *then*

$$L_{i,k}^{A_1,A_2}[f(x,y)\circledast g(x,y)](u,v)=e^{-i(d_1/2b_1)u^2}e^{-k(d_2/2b_2)v^2}F_{i,k}^{A_1,A_2}(u,v)G_{i,k}^{A_1,A_2}(u,v) \tag{14}$$

Proof.

$$L_{i,k}^{A_1,A_2}[f(x,y)\circledast g(x,y)](u,v)$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\circledast g(x,y)K_{A_1,i}(x,u)K_{A_2,k}(y,v)\,dx\,dy$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{\frac{1}{i2\pi b_1}}\sqrt{\frac{1}{k2\pi b_2}}e^{-i(a_1/2b_1)x^2}e^{-k(a_2/2b_2)y^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(a_1/2b_1)\tau^2}e^{k(a_2/2b_2)\eta^2}f(\tau,\eta)g(x-\tau,y-\eta)e^{i(a_1/2b_1)(x-\tau)^2}e^{k(a_2/2b_2)(y-\eta)^2}\,d\tau\,d\eta$$
$$\sqrt{\frac{1}{i2\pi b_1}}e^{i((d_1/2b_1)u^2+(a_1/2b_1)x^2-(1/b_1)ux)}\sqrt{\frac{1}{k2\pi b_2}}e^{k((d_2/2b_2)v^2+(a_2/2b_2)y^2-(1/b_2)vy)}\,dx\,dy$$

By making the change of variables, $s=x-\tau, t=y-\eta$, we obtain

$$L_{i,k}^{A_1,A_2}[f(x,y)\circledast g(x,y)](u,v)$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(d_1/2b_1)u^2}e^{-k(d_2/2b_2)v^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau,\eta)\sqrt{\frac{1}{i2\pi b_1}}e^{i((d_1/2b_1)u^2+(a_1/2b_1)\tau^2-(1/b_1)u\tau)}\sqrt{\frac{1}{k2\pi b_2}}e^{k((d_2/2b_2)v^2+(a_2/2b_2)\eta^2-(1/b_2)v\eta)}\,d\tau\,d\eta$$
$$g(s,t)\sqrt{\frac{1}{i2\pi b_1}}e^{i((d_1/2b_1)u^2+(a_1/2b_1)s^2-(1/b_1)us)}$$

$$\sqrt{\frac{1}{k2\pi b_2}}e^{k((d_2/2b_2)v^2+(a_2/2b_2)t^2-(1/b_2)vt)}\,ds\,dt$$
$$=e^{-i(d_1/2b_1)u^2}e^{-k(d_2/2b_2)v^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\tau,\eta)K_{A_1,i}(\tau,u)K_{A_2,k}(\eta,v)\,d\tau\,d\eta$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(s,t)K_{A_1,i}(s,u)K_{A_2,k}(t,v)\,ds\,dt$$
$$=e^{-i(d_1/2b_1)u^2}e^{-k(d_2/2b_2)v^2}F_{i,k}^{A_1,A_2}(u,v)G_{i,k}^{A_1,A_2}(u,v). \quad \square$$

Theorem 3. *For any two RB signals f(x,y) and g(x,y), the following equation holds*

$$f(x,y)\circledast g(x,y)=L_{i,k}^{A_1^{-1},A_2^{-1}}\{e^{-i(d_1/2b_1)u^2}e^{-k(d_2/2b_2)v^2}F_{i,k}^{A_1,A_2}(u,v)G_{i,k}^{A_1,A_2}(u,v)\}(x,y) \tag{15}$$

Proof. Eq. (15) can be easily obtained by Eqs. (9) and (14). □

Corollary 2. *If the parameters $A_1=A_2=(0,1,-1,0)$ in Theorems* 2 *and* 3*, then the results will be reduced to the RB convolution theorem with respect to the RBFT* [24,25].

Proof. The results can be proved from the definitions of the RBFT, RBCT, RB convolution and the RBCCV. □

5. Reduced biquaternion canonical correlation

Definition 3. The cross-RBCCR for two RB signals *f*(*x*,*y*) and *g*(*x*,*y*) is defined as follows:

$$f(x,y)\otimes g(x,y)=\sqrt{\frac{1}{i2\pi b_1}}\sqrt{\frac{1}{k2\pi b_2}}e^{-i(a_1/2b_1)x^2}e^{-k(a_2/2b_2)y^2}\langle e^{i(a_1/2b_1)x^2}e^{k(a_2/2b_2)y^2}f(x,y),g(x,y)e^{-i(a_1/2b_1)x^2}e^{-k(a_2/2b_2)y^2}\rangle \tag{16}$$

where $\langle\cdot,\cdot\rangle$ is 2-D correlation operator, that is,

$$\langle f(x,y),g(x,y)\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\tau,\eta)\overline{g(\tau-x,\eta-y)}\,d\tau\,d\eta \tag{17}$$

If *f*(*x*,*y*)=*g*(*x*,*y*), from (16) we get the auto-RBCCR:

$$f(x,y)\otimes f(x,y)=\sqrt{\frac{1}{i2\pi b_1}}\sqrt{\frac{1}{k2\pi b_2}}e^{-i(a_1/2b_1)x^2}e^{-k(a_2/2b_2)y^2}\langle e^{i(a_1/2b_1)x^2}e^{k(a_2/2b_2)y^2}f(x,y),f(x,y)e^{-i(a_1/2b_1)x^2}e^{-k(a_2/2b_2)y^2}\rangle \tag{18}$$

If the parameter $A_1=A_2=(0,1,-1,0)$, the RBCCR will be reduced to RB correlation.

Lemma 1. *For any two RB signal f(x,y) and g(x,y), the relationship between cross-RBCCR and RBCCV for f(x,y) and g(x,y) as follows*:

$$f(x,y)\otimes g(x,y)=f(x,y)\circledast\overline{g(-x,-y)} \tag{19}$$

Proof. The proof of Eq. (19) can be easily obtained from the definitions of cross-RBCCR and RBCCV. □

Theorem 4. *Let f(x,y) and g(x,y) are two RB signals, then the following equation holds*

$$L_{i,k}^{A_1,A_2}[f(x,y)\otimes g(x,y)](u,v)=e^{-i(d_1/2b_1)u^2}e^{-k(d_2/2b_2)v^2}F_{i,k}^{A_1,A_2}(u,v)G_{i,k}^{A_1',A_2'}(-u,-v) \tag{20}$$

where $A_1'=(a_1,-b_1,-c_1,d_1)$, $A_2'=(a_2,-b_2,-c_2,d_2)$.

Proof. The results can be proved from the Lemma 1 and Theorem 2. □

Theorem 5. *The RBCCR of two RB signals f(x,y) and g(x,y) can be calculated by the product of their RBCT and modulated by a chip in RBCT domain, that is*

$$f(x,y)\otimes g(x,y)=L_{i,k}^{A_1^{-1},A_2^{-1}}\{e^{-i(d_1/2b_1)u^2}e^{-k(d_2/2b_2)v^2}F_{i,k}^{A_1,A_2}(u,v)G_{i,k}^{A_1',A_2'}(-u,-v)\} \tag{21}$$

Proof. The results can be easily obtained by Eqs. (9) and (20). □

6. Fast algorithm

In this section, we give the discrete form of RBCT (DRBCT) and discuss the fast algorithm of DRBCT. The simulation of using RBCT to transform color images is also presented.

6.1. The discrete form of RBCT

The discrete algorithm we used here is similar with the fractional Fourier transform [12]. To derive the DRBCT, we first sample the RB signal *f*(*x*,*y*) and the output function $F_{i,k}^{A_1,A_2}(u,v)$ by the interval $\Delta x,\Delta y,\Delta u,\Delta v$ as

$$f(m_1,m_2)=f(m_1\cdot\Delta x,m_2\cdot\Delta y),\quad F_{i,k}^{A_1,A_2}(n_1,n_2)=F_{i,k}^{A_1,A_2}(n_1\cdot\Delta u,n_2\cdot\Delta v) \tag{22}$$

where $m_1, m_2=-M,-M+1,\dots,M$; $n_1, n_2=-N,-N+1,\dots,N$.

From Eq. (6), we can give the DRBCT as follows:

$$F_{i,k}^{A_1,A_2}(n_1,n_2)\triangleq L_{i,k}^{A_1,A_2}[f(m_1,m_2)](n_1,n_2)=\begin{cases}\sum_{m_1=-M}^{M}\sum_{m_2=-M}^{M}f(m_1,m_2)K_i^{A_1}(m_1,n_1)K_k^{A_2}(m_2,n_2), & b_1b_2\neq 0\\ \sqrt{d_1d_2}f(d_1n_1,d_2n_2)e^{i(c_1d_1/2)(n_1\Delta u)^2}e^{k(c_2d_2/2)(n_2\Delta v)^2}, & b_1b_2=0\end{cases} \tag{23}$$

where

$$K_i^{A_1}(m_1,n_1)=\sqrt{\frac{1}{i2\pi b_1}}e^{i[(d_1/2b_1)(n_1\Delta u)^2+(a_1/2b_1)(m_1\Delta x)^2-(1/b_1)m_1n_1\Delta u\Delta x]} \tag{24}$$

$$K_k^{A_2}(m_2,n_2)=\sqrt{\frac{1}{k2\pi b_2}}e^{k[(d_2/2b_2)(n_2\Delta v)^2+(a_2/2b_2)(m_2\Delta y)^2-(1/b_2)m_2n_2\Delta v\Delta y]} \tag{25}$$

Here, we only discuss the condition $b_1b_2\neq 0$. We need to find the conditions that the RB signal $f(m_1, m_2)$ could be reconstructed from the DRBCT $F_{i,k}^{A_1,A_2}(n_1,n_2)$. We rewritten

Eq. (9) in discrete form

$$f(m_1,m_2)=\sum_{n_1=-N}^{N}\sum_{n_2=-N}^{N}F_{i,k}^{A_1,A_2}(n_1,n_2)K_i^{A_1^{-1}}(n_1,m_1)K_k^{A_2^{-1}}(n_2,m_2)$$

$$=\sum_{n_1=-N}^{N}\sum_{n_2=-N}^{N}\left[\sum_{s_1=-M}^{M}\sum_{s_2=-M}^{M}f(s_1,s_2)K_i^{A_1}(s_1,n_1)K_k^{A_2}(s_2,n_2)\right]$$

$$K_i^{A_1^{-1}}(n_1,m_1)K_k^{A_2^{-1}}(n_2,m_2)$$

$$=\sum_{s_1=-M}^{M}\sum_{s_2=-M}^{M}f(s_1,s_2)\left[\sum_{n_1=-N}^{N}K_i^{A_1}(s_1,n_1)K_i^{A_1^{-1}}(n_1,m_1)\right]$$

$$\left[\sum_{n_2=-N}^{N}K_k^{A_2}(s_2,n_2)K_k^{A_2^{-1}}(n_2,m_2)\right] \tag{26}$$

where

$$\sum_{n_1=-N}^{N}K_i^{A_1}(s_1,n_1)K_i^{A_1^{-1}}(n_1,m_1)$$

$$=\sum_{n_1=-N}^{N}\frac{1}{2\pi|b_1|}e^{[(a_1/2b_1)((s_1\Delta x)^2-(m_1\Delta x)^2)-(1/b_1)n_1(s_1-m_1)\Delta x\Delta u]} \tag{27}$$

$$\sum_{n_2=-N}^{N}K_k^{A_2}(s_2,n_2)K_k^{A_2^{-1}}(n_2,m_2)$$

$$=\sum_{n_2=-N}^{N}\frac{1}{2\pi|b_2|}e^{[(a_2/2b_2)((s_2\Delta y)^2-(m_2\Delta y)^2)-(1/b_2)n_2(s_2-m_2)\Delta y\Delta v]} \tag{28}$$

If we want the summations for n_1 and n_2 in Eqs. (27) and (28), respectively, to be $\delta(s_1-m_1)$ and $\delta(s_2-m_2)$, then

$$\Delta x\Delta u=\frac{2\pi|b_1|}{2N+1},\quad \Delta y\Delta v=\frac{2\pi|b_2|}{2N+1} \tag{29}$$

Substitute Eq. (29) into Eqs. (27) and (28), we have

$$\sum_{n_1=-N}^{N}K_i^{A_1}(s_1,n_1)K_i^{A_1^{-1}}(n_1,l_1)=\frac{2N+1}{2\pi|b_1|}\delta(s_1-m_1) \tag{30}$$

$$\sum_{n_2=-N}^{N}K_k^{A_2}(s_2,n_2)K_k^{A_2^{-1}}(n_2,l_2)=\frac{2N+1}{2\pi|b_2|}\delta(s_2-m_2) \tag{31}$$

In order for Eq. (26) to hold, we normalize Eqs. (30) and (31) by the factor $2\pi|b_1|/(2N+1)$ and $2\pi|b_2|/(2N+1)$, respectively.

In the end, we have the discrete forms of $K_i^{A_1}(m_1,n_1)$ and $K_k^{A_2}(m_2,n_2)$:

$$K_i^{A_1}(m_1,n_1)$$

$$=\sqrt{\frac{1}{2N+1}}e^{i[(d_1/2b_1)(n_1\Delta u)^2+(a_1/2b_1)(m_1\Delta x)^2-(2\pi\,\mathrm{sgn}(b_1)m_1n_1/(2N+1))]} \tag{32}$$

$$K_k^{A_2}(m_2,n_2)$$

$$=\sqrt{\frac{1}{2N+1}}e^{k[(d_2/2b_2)(n_2\Delta v)^2+(a_2/2b_2)(m_2\Delta y)^2-(2\pi\,\mathrm{sgn}(b_2)m_2n_2/(2N+1))]} \tag{33}$$

where $sgn(b)$ equals 1 when $b\geq 0$ and -1 when $b<0$.

6.2. Fast algorithm of DRBCT

In [24], the implementation of RBFT by two complex 2-D FT, that is, for RB signal $f(x,y)=f_{1+2}(x,y)e_1+f_{1-2}(x,y)e_2$, we have

$$RBFT[f(x,y)](u,v)=F_{1+2}(u,v)e_1+F_{1-2}(u,v)e_2 \tag{34}$$

where $F_{1+2}(u,v)$,$F_{1-2}(u,v)$ are the complex 2-D FT of $f_{1+2}(x,y)$ and $f_{1-2}(x,y)$, respectively.

The DRBCT of a RB signal $f(m_1,m_2)$ can be rewritten as follows:

$$F_{i,k}^{A_1,A_2}(n_1,n_2)=\frac{1}{2N+1}e^{i[(d_1/2b_1)(n_1\Delta u)^2]}e^{k[(d_2/2b_2)(n_2\Delta v)^2]}\sum_{m_1=-M}^{M}\sum_{m_2=-M}^{M}f(m_1,m_2)$$

$$e^{i[(a_1/2b_1)(m_1\Delta x)^2]}e^{k[(a_2/2b_2)(m_2\Delta y)^2]}e^{-i(2\pi\,\mathrm{sgn}(b_1)m_1n_1/(2N+1))}$$

$$e^{-k(2\pi\,\mathrm{sgn}(b_2)m_2n_2/(2N+1))} \tag{35}$$

Let

$$g(m_1,m_2)=f(m_1,m_2)e^{i[(a_1/2b_1)(m_1\Delta x)^2]}e^{k[(a_2/2b_2)(m_2\Delta y)^2]} \tag{36}$$

then

$$F_{i,k}^{A_1,A_2}(n_1,n_2)=\frac{1}{2N+1}e^{i[(d_1/2b_1)(n_1\Delta u)^2]}e^{k[(d_2/2b_2)(n_2\Delta v)^2]}$$

$$\sum_{m_1=-M}^{M}\sum_{m_2=-M}^{M}g(m_1,m_2)$$

$$\times e^{-i(2\pi\,\mathrm{sgn}(b_1)m_1n_1/(2N+1))}e^{-k(2\pi\,\mathrm{sgn}(b_2)m_2n_2/(2N+1))}$$

$$=\frac{1}{2N+1}e^{i[(d_1/2b_1)(n_1\Delta u)^2]}e^{k[(d_2/2b_2)(n_2\Delta v)^2]}$$

$$\mathrm{RBFT}[g(m_1,m_2)](\mathrm{sgn}(b_1)n_1,\mathrm{sgn}(b_2)n_2) \tag{37}$$

We can implement the RBCT by the RBFT, the calculation steps are as follows:

Step (1): Calculate $g(m_1, m_2)$ from $f(m_1, m_2)$ using Eq. (36);
Step (2): Decomposition $g(m_1, m_2)$ into e_1-e_2 form, i.e., $g(m_1,m_2)=g_{1+2}(m_1, m_2)e_1+g_{1-2}(m_1, m_2)e_2$;
Step (3): Calculate 2-D FT of $g_{1+2}(m_1, m_2)$ and $g_{1-2}(m_1, m_2)$, respectively, get the results: $G_{1+2}(n_1, n_2)$, $G_{1-2}(n_1,n_2)$;
Step (4): Calculate $F_{i,k}^{A_1,A_2}(n_1,n_2)$ by following equations:

$$F_{i,k}^{A_1,A_2}(n_1,n_2)=\frac{1}{2N+1}e^{i[(d_1/2b_1)(n_1\Delta u)^2]}e^{k[(d_2/2b_2)(n_2\Delta v)^2]}$$

$$[G_{1+2}(n_1,n_2)e_1+G_{1-2}(n_1,n_2)e_2] \tag{38}$$

Finally, we can calculate the RBCT through two complex 2D FFT. Therefore, the computational complexity of the RBCT is similar to FFT.

6.3. Simulations

The simulation of using DRBCT to transform color images is presented in this subsection. We encode the three channel components of RGB image on the three imaginary parts of a pure reduced biquaternion. In other words, a pixel at image coordinate (m,n) in an RGB image

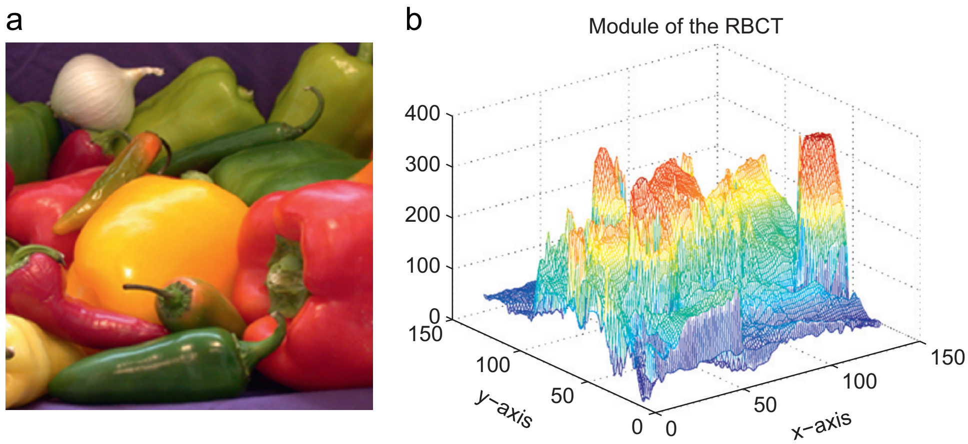

Fig. 1. The RBCT of color image. (a) the color images; (b) the module of the RBCT spectrum. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

can be represented as

$$f(m,n)=f_R(m,n)i+f_G(m,n)j+f_B(m,n)k \tag{39}$$

where $f_R(m,n)$,$f_G(m,n)$ and $f_B(m,n)$ are the red, green and blue components of the pixel, respectively.

In the experiment, we set the transformation parameters as follows: $A_1=(\sqrt{129},128/2\pi,2\pi,\sqrt{129})$, $A_2=(10,128/4\pi,4\pi,12.9)$. In Fig. 1, (a) is the original color images, 128×128 pixels. The module of RBCT spectrum is presented in Fig. 1, (b).

7. Conclusion

In this research, we proposed the forward and inverse transforms of the RBCT, which can be seen as the generalization of traditional RBFT. The RBCT can not only process RB signals but also real scalar or complex signals. So, the RBCT is also the generalization of traditional LCT. In addition, the Parseval's theorem associated with RBCT is developed. Moreover, the RBCCV and RBCCR are defined, the convolution and correlation theorem related to RBCT are discussed. Finally, the Fast algorithm of RBCT is presented in this paper. Our future work will be focused on the use of RBCT in color image processing.

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