Developed method of excess fractions for calibrating the effective optical thickness of a Fabry–Perot étalon

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A general method of calculating the integral-order number of interference is developed, completely without a trial method. The developed method of excess fractions is extended to the calibration of the effective optical thickness of a Fabry–Perot étalon with the disperion of the refractive index and the phase shift on reflection. Several useful methods of calibration are provided and illustrated by examples. The condition under which the phase shift can be neglected is given for calculating the correct integral-order difference between wavelengths. A possible method of precise measurement of the dispersive phase shift is given. It is pointed out that exact knowledge of the integral order is not even necessary and that the phase shift can thus essentially be neglected for a precise measurement of the wavelength as long as the effective optical thickness is calibrated as the product of the fringe order of interference and the standard wavelength.

INTRODUCTION

In the application of a Fabry–Perot étalon to the comparison and measurement of wavelengths, precise calibration of the effective optical thickness of the étalon is of fundamental importance. In practice, one often uses the étalon made by a solid glass plate or a fused quartz plate with metallic or dielectric reflecting films or those made by two reflecting plates separated by a air gap; hence the calibration of the étalons is complicated by the dispersion of the refractive index and the phase shift on reflection. The traditional methods of exact fractions[1] and excess fractions[2] that can be used to determine the integral orders of interference are inconvenient for a systematic study of these problems because of their trial feature. The modified method of excess fractions[3] provides a new means of studying these problems. However, it was used previously only for studying the case in which the phase shift is neglected in vacuum, and a method was not developed for selecting a single z value for a combination of wavelengths satisfying the requirement that $1 < |4(1/\lambda_2 - 1/\lambda_1)\Delta d'| \leq N$ (integer $N > 1$); that is, a comparison was still needed for finding the correct integral order.

In this paper, the modified method of excess fractions is developed further. A new technique of calculating the integral orders is given for a general combination of wavelengths. The developed method of excess fractions is extended to the calibration of the effective optical thickness of a Fabry–Perot étalon with the dispersion of the refractive index and the phase shift on reflection. Several useful methods of calibration are provided and illustrated by examples. It is pointed out that exact knowledge of the integral order is not even necessary and that the phase shift can thus essentially be neglected for a precise measurement of wavelength as long as the requirements are satisfied.

GENERAL OPERATION

For normal incidence, the path difference between successive rays emerging from a Fabry–Perot étalon is

$$(m + e)\lambda = 2nd + \frac{\phi\lambda}{\pi}, \tag{1}$$

where m is the integral order of the innermost bright fringe, e is the fractional order at the center, λ is the wavelength, n is the refractive index of the étalon, d is the étalon thickness, and ϕ is the phase shift on internal reflection. For the sake of convenience Eq. (1) may be rewritten in the following forms:

$$\begin{aligned}(m + e)\lambda &= 2nd(1 + \alpha)\\ &= 2n(\lambda)d\\ &= 2nd(\lambda),\end{aligned} \tag{2}$$

where

$$\alpha = \frac{\phi\lambda}{2\pi nd},$$

$$n(\lambda) = n(1 + \alpha) = n + \frac{\phi\lambda}{2\pi d}, \tag{3}$$

and

$$d(\lambda) = d(1 + \alpha) = d + \frac{\phi\lambda}{2\pi n}.$$

For the sake of simplicity $|\phi|$ may be considered to be less than π.[4] For the conditions under consideration, d is so large and λ is so small that $\alpha \ll 1$, $n \gg \phi\lambda/2\pi d$, and $d \gg \phi\lambda/2\pi n$. Taking the derivative of $d(\lambda)$ with respect to λ yields

$$\frac{\Delta d(\lambda)}{\Delta\lambda} = \frac{\Delta\left(\dfrac{\phi\lambda}{2\pi n}\right)}{\Delta\lambda} = C_\phi. \tag{4}$$

It is well known that metallic reflecting films, such as Al and Ag, produce a phase shift that varies so that the product $\phi\lambda$ is nearly constant[5]; i.e., C_ϕ is almost equal to zero. For dielectric multilayers, however, the product $\phi\lambda$ may vary much more rapidly with wavelength than does that for metals. The phenomenon must be reckoned with in the calibra-

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tion of the effective optical thickness of the ĕtalon. Under the condition of linear approximation, which is in good agreement with experiment,[6,7] C_ϕ is almost constant. If $\phi_0 = 0$ when $\lambda = \lambda_0$, then from the integral of Eq. (4) we can get

$$\frac{\phi\lambda}{\pi} = 2nC_\phi(\lambda - \lambda_0). \quad (5)$$

Let λ_s and λ_u be the known precise wavelengths; then from Eq. (2) we have

$$(m_s + e_s)\lambda_s = (m_u + e_u)\frac{n(\lambda_s)\lambda_u}{n(\lambda_u)}, \quad (6)$$

where m_s and m_u are the integral orders of the first bright rings, e_s and e_u are the fractional orders at the center, and $n(\lambda_s)$ and $n(\lambda_u)$ are the effective refractive indexes corresponding to λ_s and λ_u. For each wavelength, the fractional order may be obtained from measurements of the bright ring diameters. When $n(\lambda_s)$ and $n(\lambda_u)$ are not precise enough, we may consider $n(\lambda_s)\lambda_u/n(\lambda_u)$ an effective unknown wavelength. Therefore the above-described problem is essentially a matter of comparison of the unknown wavelength with the standard wavelength. The integers m_s and m_u can be found by the developed method of excess fractions as follows.

If we put $d = d' + \Delta d'$, $n_i = n_i' + \Delta n_i'$, $e_i = e_i' + \Delta e_i'$, $C_\phi = C_\phi' + \Delta C_\phi'$, $\lambda_0 = \lambda_0' + \Delta\lambda_0'$, and $\lambda_u = \lambda_u' + \Delta\lambda_u'$, where the subscript $i = s,u$ and where

$$\lambda_u = \frac{n(\lambda_s)}{n(\lambda_u)}\lambda_u, \qquad \lambda_u' = \frac{n'(\lambda_s)}{n'(\lambda_u)}\lambda_u,$$

$$\Delta\lambda_u' = \left[\frac{\Delta n'(\lambda_s)}{n'(\lambda_s)} - \frac{\Delta n'(\lambda_u)}{n'(\lambda_u)}\right]\lambda_u',$$

then from Eq. (6) we have

$$m_u + e_u' = (m_s + e_s')\frac{\lambda_s}{\lambda_u'} + \delta, \quad (7)$$

where

$$\delta = -(m_u + e_u)\frac{\Delta\lambda_u'}{\lambda_u'} + \frac{\lambda_s}{\lambda_u'}\Delta e_s' - \Delta e_u'. \quad (8)$$

By using Eqs. (3) and (5) and $(m_u + e_u) \approx 2n'(\lambda_u)d'/\lambda_u$, we can write Eq. (8) as

$$\delta = \delta_e + \delta_n + \delta_\phi, \quad (8a)$$

where

$$\delta_e = \frac{n_u'[1 + C_\phi'(\lambda_u - \lambda_0')/d']\lambda_s}{n_s'[1 + C_\phi'(\lambda_s - \lambda_0')/d']\lambda_u}\Delta e_s' - \Delta e_u',$$

$$\delta_n = \frac{2d'(n_s'\Delta n_u' - n_u'\Delta n_s')[1 + C_\phi'(\lambda_u - \lambda_0')/d']}{n_s'\lambda_u},$$

and

$$\delta_\phi = 2n_u'\left(1 - \frac{\lambda_s}{\lambda_u}\right)\frac{d'\Delta C_\phi' - C_\phi'\Delta d' + C_\phi'^2\Delta\lambda_0'}{d' + C_\phi'(\lambda_s - \lambda_0')}.$$

If we express the initial approximate quantities in terms of the uncertainty as $d' \pm \Delta d$, $n_i' \pm \Delta n_i$, $e_i' \pm \Delta e_i$, $C_\phi' \pm \Delta C_\phi$, and $\lambda_0' \pm \Delta\lambda_0$, we may estimate the uncertainty of δ as

$$\delta_e = \left(1 + \frac{\lambda_s}{\lambda_u}\right)\Delta e, \qquad \delta_n = \frac{4d'\Delta n}{\lambda_u},$$

$$\delta_\phi = 2n_u'\left|1 - \frac{\lambda_s}{\lambda_u}\right|\Delta C_\phi, \quad (8b)$$

where the relations $\Delta\lambda \ll \lambda \ll \Delta d \ll d'$, $\alpha_u' \ll 1$, $n_s' = n_u'$, $\Delta n_s = \Delta n_u = \Delta n$, and $\Delta e_s = \Delta e_u = \Delta e$ have been used. An approximate value of the effective optical thickness of the ĕtalon for λ_s is

$$n'(\lambda_s)d' \pm \Delta[n'(\lambda_s)d'], \quad (9)$$

where

$$n'(\lambda_s)d' = n_s'd' + n_s'C_\phi'(\lambda_s - \lambda_0')$$

and

$$\Delta[n'(\lambda_s)d'] = \Delta nd' + n_s'\Delta d + |\lambda_s - \lambda_0'|(\Delta n|C_\phi'| + \Delta C_\phi n_s') + n_s'|C_\phi'|\Delta\lambda_0.$$

According to the modified method of excess fractions,[3] for example, we take the integer

$$m_s' = \mathrm{Int.P}\left(\frac{2\{n'(\lambda_s)d' + \Delta[n'(\lambda_s)d']\}}{\lambda_s}\right) \quad (10)$$

as an approximate value of m_s, where Int.P( ) denotes the integral part of the expression within the parentheses, and we may write

$$m_s = m_s' + {}^s x, \quad (11)$$

where ${}^s x$ is an unknown integer and

$$-\frac{4\Delta[n'(\lambda_s)d']}{\lambda_s} = {}^s x_m \leq {}^s x \leq 0. \quad (12)$$

An approximate order for λ_u is calculated as

$$m_u' + e_u'' = (m_s' + e_s')\frac{\lambda_s}{\lambda_u'}. \quad (13)$$

From Eqs. (7), (11), and (13) we can obtain the following relation:

$$m_u + e_u' = (m_u' + {}^s x - z) + \left[e_u'' + z + {}^s x\left(\frac{\lambda_s}{\lambda_u'} - 1\right)\right] + \delta, \quad (14)$$

where z is an integer that can be selected according to the specific rules (see below).

For simplicity, we write

$$A_u{}^s = \frac{\lambda_s}{\lambda_u'} - 1 = \frac{n_u'[1 + C_\phi'(\lambda_u - \lambda_0')/d']\lambda_s}{n_s'[1 + C_\phi'(\lambda_s - \lambda_0')/d']\lambda_u} - 1. \quad (15)$$

If the combination of λ_s and λ_u satisfies the requirement that

$$|A_u{}^s \cdot {}^s x_m| \leq 1, \quad (16)$$

then from Eq. (14) we make

$${}^s x = \mathrm{Int}\left(\frac{e_u' - e_u'' - z}{A_u{}^s}\right), \quad (17)$$

where Int( ) denotes the integer nearest the value of the expression in parentheses, and the value of z may be selected according to the rules given in Ref. 3. From Eq. (14) we obtain

$$m_u = m_u' + {}^s x - z. \tag{18}$$

From Eqs. (14) and (8), we may see that the value of ${}^s x$ found by Eq. (17) has an error

$$\Delta^s x = \frac{-\delta}{A_u^{\,s}} = \Delta x_e + \Delta x_n + \Delta x_\phi, \tag{19}$$

where

$$\Delta x_e = \frac{-\delta_e}{A_u^{\,s}}, \qquad \Delta x_n = \frac{-\delta_n}{A_u^{\,s}}, \qquad \Delta x_\phi = \frac{-\delta_\phi}{A_u^{\,s}}.$$

Using Eq. (8b) we may estimate the uncertainty of ${}^s x$ found by Eq. (17) as follows:

$$\Delta x_e = \frac{\Delta e}{|A_u^{\,s}|}\left(1 + \frac{\lambda_s}{\lambda_u}\right), \qquad \Delta x_n = \frac{4d'\Delta n}{\lambda_u |A_u^{\,s}|},$$

$$\Delta x_\phi = 2n_u'\Delta C_\phi, \tag{19a}$$

where the relation $|1 - \lambda_s/\lambda_u| \approx |A_u^{\,s}|$ has been used.

The value of ${}^s x$ found by Eq. (17) is correct, provided that $\Delta\, {}^s x < 0.5$. Thus the correct values of m_s and m_u can be found by Eqs. (11) and (18). If $\Delta x_e < 0.5$, then the error of ${}^s x$ introduced by the fractional-order errors in the calculation of Eq. (17) will be eliminated. Hence the requirement that $\Delta\, {}^s x < 0.5$ can be relaxed to that of $\Delta x_e < 0.5$ and $\Delta x_n + \Delta x_\phi < 0.5$. According to the requirements of $\Delta x_e < 0.5$ and $\Delta x_n + \Delta x_\phi < 0.5$, from Eq. (19a) we can get

$$|A_u^{\,s}| > A_e = 2\Delta e\left(1 + \frac{\lambda_s}{\lambda_u}\right)$$

and

$$|A_u^{\,s}| > A_{n\phi} = \frac{4d'\Delta n}{\lambda_u(0.5 - 2n_u'\Delta C_\phi)}. \tag{19b}$$

If the requirements that $\Delta x_e < 0.5$ and $\Delta x_n + \Delta x_\phi < 0.5$ are not satisfied, then the value of ${}^s x$ found by Eq. (17) will have an error; however, the integral-order difference $\Delta m_{s,u}$ obtained by subtracting m_u of Eq. (18) from m_s of Eq. (11) can be correct, and we have

$$\Delta m_{s,u} = m_s - m_u = m_s' - m_u' + z, \tag{20}$$

provided that $\Delta m_{s,u}$ can be determined unambiguously. In fact, from Eq. (2) we can obtain

$$\Delta m_{s,u} = 2d\left[\frac{n(\lambda_s)}{\lambda_s} - \frac{n(\lambda_u)}{\lambda_u}\right] + e_u - e_s,$$

where $\Delta m_{s,u}$ can be determined unambiguously if its uncertainty is less than 0.5. This implies that

$$2\Delta d\left[\frac{n'(\lambda_s)}{\lambda_s} - \frac{n'(\lambda_u)}{\lambda_u}\right] + 2d'\left[\frac{\Delta n(\lambda_s)}{\lambda_s} - \frac{\Delta n(\lambda_u)}{\lambda_u}\right]$$

$$+ \Delta e_u - \Delta e_s = \frac{1}{2}({}^s x_m + 2\Delta e_s')A_u^{\,s} - \delta < 0.5;$$

hence we get

$$(|{}^s x_m| + 2\Delta e)|A_u^{\,s}| + 2|\delta| < 1,$$

i.e.,

$$|A_u^{\,s}| < Am = \frac{1 - 2|\delta|}{|{}^s x_m| + 2\Delta e}. \tag{21}$$

Obviously, under condition (21), the correct $\Delta m_{s,u}$ can be obtained by using Eq. (20). Because Am always satisfies the requirement of relation (16), the single z can be chosen.

We know from Eqs. (19a) that, when $\Delta x_e < 0.5$ and $\Delta x_n + \Delta x_\phi < 0.5$, $A_u^{\,s}$ may not satisfy condition (16) but it can satisfy the following requirement:

$$1 < |A_u^{\,s} \cdot {}^s x_m| \le N, \tag{22}$$

where N is an integer and $N > 1$. Under this condition, the N different values of ${}^s x$ satisfying the requirement that $|{}^s x| \le |{}^s x_m|$ can be obtained from Eq. (17) for the N possible values of z. How will we select the single z corresponding to the correct ${}^s x$? For this purpose we may take the wavelengths $\lambda_1, \lambda_2, \ldots, \lambda_p$ between λ_s and λ_u and make each combination of the adjacent wavelengths satisfy condition (21); thus the correct $\Delta m_{s,1}, \Delta m_{1,2}, \ldots, \Delta m_{p,u}$ can be found by using Eq. (20). We can then add them together to obtain the correct $\Delta m_{s,u}$ between λ_s and λ_u as

$$\Delta m_{s,u} = \Delta m_{s,1} + \Delta m_{1,2} + \Delta m_{2,3} + \ldots + \Delta m_{p,u}. \tag{23}$$

By substituting the obtained $\Delta m_{s,u}$ into the equation [which was obtained from Eq. (20)]

$$z = \Delta m_{s,u} - m_s' + m_u', \tag{24}$$

we can find the correct z. By substituting the obtained z into Eq. (17), we can find the correct ${}^s x$, provided that $A_u^{\,s}$ satisfies the requirements that $\Delta x_e < 0.5$ and $\Delta x_n + \Delta x_\phi < 0.5$, i.e., relation (19b). After the correct ${}^s x$ is obtained, the correct m_s and m_u can be found from Eqs. (11) and (18). Finally, the effective optical thickness $n(\lambda)d$ can be obtained from Eq. (2) as the product of the order of interference and the standard wavelength.

It can be seen from Eqs. (19a) that the values of Δx_e and Δx_n are inversely proportional to $A_u^{\,s}$ and that Δx_ϕ is practically independent of $A_u^{\,s}$; moreover, Δx_e is dependent on only Δe, Δx_n is dependent only on Δn, and Δx_ϕ is dependent only on ΔC_ϕ in the light of uncertainty. Therefore the integral-order errors introduced by the fractional-order error, the refractive-index error, and the phase-shift error can be discussed separately. Practically, under the condition that the phase shift of reflection be neglected, we may first find the integral order m'' according to the requirements of $\Delta x_e < 0.5$ and $\Delta x_n < 0.5$. We then find the integral-order contribution m_ϕ of the phase shift with the known C_ϕ'. In fact, neglecting the phase shift implies that $\phi = 0$, i.e., that $C_\phi' = 0$ and $\Delta C_\phi = C_\phi$; from Eqs. (19a) we obtain $\Delta x_{\phi=0} = 2n_u'C_\phi$ and can unambiguously determine that

$$m_\phi = \mathrm{Int}(2n_u'C_\phi'), \tag{25}$$

provided that the uncertainty of the initial C_ϕ' is

$$\Delta C_\phi < \frac{1}{2n_u'}[0.5 - |\mathrm{Int}(\Delta x_{\phi=0}) - \Delta x_{\phi=0}|]. \tag{25a}$$

Hence we obtain the correct integral order,

$$m = m'' + m_\phi. \tag{26}$$

This is a practical method that has several advantages over the aforementioned general method. First, the calculation by the practical method is simpler. Second, the requirements for the errors of the refractive index and the phase shift can be relaxed for determining the correct m for the following reason. In the general method the errors of the refractive index and the phase shift are taken together into account, and $\Delta x_e < 0.5$ and $\Delta x_n + \Delta x_\phi < 0.5$ must be satisfied for the correct m to be found, but the practical method requires only that $\Delta x_e < 0.5$ and $\Delta x_n < 0.5$ and that relation (25a) hold. It should be pointed out that the proper intervals of wavelengths must still be chosen by relation (21).

It should be noted that even if the phase shift is neglected the requirements for determining the correct m often are not satisfied because of the larger values of d, Δd, Δn, and ΔC_ϕ or the smaller range of usable wavelengths that occurs in a practical calibration. Hence we can obtain only an approximate integral order m''' by Eq. (17), and we may write

$$m = m''' + \Delta m, \tag{27}$$

where Δm is the integral-order error of m''', which might contain the errors introduced by Δe and Δn and by neglecting the phase shift. Now we may obtain the assumed optical thickness

$$(m''' + e)\lambda = 2nd'''(\lambda). \tag{28}$$

Using Eqs. (27) and (2), we can obtain

$$d'''(\lambda) = d(\lambda) - \frac{\Delta m \lambda}{2n};$$

hence

$$\frac{\Delta d'''(\lambda)}{\Delta\lambda} = C_\phi''' = C_\phi - \frac{\Delta m}{2n} + \frac{\Delta m \lambda}{2n^2}\frac{\Delta n}{\Delta \lambda}.$$

In consideration of the dimension of dispersion of the refractive index in the practical materials, the last term can be neglected; thus we have

$$C_\phi''' = C_\phi - \frac{\Delta m}{2n},$$

i.e.,

$$\Delta m = \mathrm{Int}[2n(C_\phi - C_\phi''')]. \tag{29}$$

The correct value of Δm can be determined by Eq. (29), provided that the error of calculating Δm is less than 0.5. This implies that

$$\Delta C_\phi + \Delta C_\phi''' < \frac{1}{4n}. \tag{30}$$

It should be pointed out that the value of C_ϕ''' can be obtained by experiment. In fact, once an approximate m''' is determined, we have

$$d'''(\lambda) = \frac{(m''' + e)\lambda}{2n'}, \tag{31}$$

where n' is the initial approximate value of the refractive index. The variation of the obtained $d'''(\lambda)$ can be fitted well to a first-order polynomial,

$$d'''(\lambda) = \sum_{j=0}^{1} d_j \lambda^j, \tag{32}$$

where the coefficients d_j can be obtained from a least-squares fit. The coefficient d_1 of the first-order term is simply C_ϕ''', i.e.,

$$C_\phi''' = d_1.$$

This method of finding the integral orders m from an approximate m''' is called the correctional method.

Clearly, if the obtained integral orders are correct, i.e., if $\Delta m = 0$, then we have $C_\phi = d_1$; hence this is a possible method of precisely measuring C_ϕ.

NUMERICAL EXAMPLES

In order to show the manner of operation of the present method, we shall use a numerical example. Let us assume that the correct m_i and e_i for the known wavelengths λ_i are as shown in Table 1. These data are obtained from Eqs. (1) and (5) under the following assumption. For a given étalon, the precise value of the étalon thickness is $d = 2.129374582$ mm, the precise refractive indices n_i are given in Table 1, the dielectric multilayer films have a linear phase-shift range between 0.54 and 0.66 μm, $\lambda_0 = 0.59791304$ μm, and $C_\phi = -0.6454725525$.

Our task is to find the correct integral orders m_i with the developed method of excess fractions according to initial approximate values and then to calibrate the effective optical thickness of the given étalon. For the purpose of specification we shall describe several useful methods as follows.

The General Method

The errors of the refractive index and the phase shift are considered together in the general method.

Consider an approximate value of the étalon thickness $d' \pm \Delta d = 2.131 \pm 0.005$ mm; the measured values of e_i' and n_i' are given in Table 2, $\Delta n = 5 \times 10^{-6}$, $\Delta e = 0.01$, $C_\phi' \pm \Delta C_\phi = -0.62 \pm 0.03$, $\lambda_0' \pm \Delta\lambda_0 = 0.5979 \pm 10^{-4}$ μm, and the spectral range of calibration is from 0.54811314 to 0.65213641 μm.

First, since we must determine the correct z and $\Delta m_{s,u}$, we use condition (21) to choose the proper intervals of wavelengths in the given spectral range of calibration.

Let $\lambda_s = \lambda_1 = 0.54811314$ μm; from expression (9) we obtain $n'(\lambda_s)d' = n'(\lambda_1)d' = 3111.275232$ μm and $\Delta[n'(\lambda_s)d'] = \Delta[n'(\lambda_1)d'] = 7.312856316$ μm; from relation (12), $|{}^s x_m| =$

Table 1. Correct Orders of Interference m_i and e_i for the Known Wavelengths

i	λ_i(Å)	n_i	m_i	e_i
1	5481.1314	1.459990847	11344	0.06056
2	5561.4231	1.459655796	11177	0.69048
3	5641.6431	1.459334149	11016	0.29712
4	5721.6534	1.459025605	10859	0.92480
5	5801.2346	1.458730140	10708	0.75372
6	5881.3471	1.458443537	10560	0.78394
7	5961.3796	1.458167449	10417	0.00606
8	6041.3215	1.457901292	10277	0.26169
9	6121.1367	1.457644608	10141	0.44322
10	6201.0134	1.457396274	10009	0.07919
11	6281.1012	1.457155394	9879	0.800237
12	6361.0744	1.456922521	9754	0.005916
13	6441.1166	1.456696698	9631	0.278791
14	6521.3641	1.456477186	9511	0.306149

Table 2. Measured e_i' and n_i' Values for the Known Wavelengths

i	λ_i(Å)	n_i'	e_i'
1	5481.1314	1.459986	0.06
2	5561.4231	1.459658	0.69
3	5641.6431	1.459332	0.29
4	5721.6534	1.459029	0.93
5	5801.2346	1.458726	0.75
6	5881.3471	1.458442	0.79
7	5961.3796	1.458169	0.01
8	6041.3215	1.457901	0.27
9	6121.1367	1.457642	0.44
10	6201.0134	1.457399	0.07
11	6281.1012	1.457153	0.80
12	6361.0744	1.456924	0.01
13	6441.1166	1.456693	0.28
14	6521.3641	1.456482	0.31

$|{}^1x_m| = 53.37$; from Eq. (10), $m_s' = m_1' = \text{Int}(11379.359) = 11379$.

If $\lambda_u = \lambda_2 = 0.55614231$ μm is chosen, then from Eq. (15) we obtain $A_u^s = A_2^1 = -0.014660976$; from Eqs. (8a) and (8b), $\delta = \delta_e + \delta_n + \delta_\phi = 0.0199 + 0.0766 + 0.0013 = 0.0978$; from relation (21), $Am = 0.0151$. Now, $|A_2^1 \cdot {}^1x_m| = 0.78 < 1$, $|A_2^1| = 0.014660976 < Am$; obviously, conditions (16) and (21) have been satisfied. From Eqs. (13) and (15) we have

$$m_u' + e_u'' = (m_s' + e_s')(1 + A_u^s), \tag{13a}$$

and we may obtain $m_u' = m_2' = 11212$ and $e_u'' = e_2'' = 0.232$.

According to the selection rules of z (see Ref. 3), $A_2^1 \cdot {}^1x_m > 0$, $e_u' = e_2' = 0.69 > e_u'' = e_2'' = 0.232$, thus we select $z = 0$. From Eq. (20) we obtain $\Delta m_{s,u} = \Delta m_{1,2} = 11379 - 11212 + 0 = 167$.

In a similar manner we can obtain $\Delta m_{2,3} = 161$ for the combination of $\lambda_s = \lambda_2$ and $\lambda_u = \lambda_3 = 0.56416431$ μm; $\Delta m_{p,u} = \Delta m_{13,14} = 120$ for $\lambda_s = \lambda_{13} = 0.64411166$ μm and $\lambda_u = \lambda_{14} = 0.65213641$ μm, as shown in Table 3. The integral-order differences between the nonadjacent wavlengths can be obtained by using Eq. (23).

Second, to find the correct ${}^sx$ by Eq. (17), i.e., to meet the requirements that $\Delta x_e < 0.5$ and $\Delta x_n + \Delta x_\phi < 0.5$, we use condition (19b) to choose the proper combination of wavelengths.

If $\lambda_s = \lambda_1 = 0.54811314$ μm and $\lambda_u = \lambda_{14} = 0.65213641$ μm are chosen, then from Eq. (15) we have $A_{14}^1 = -0.161554087$; from condition (19b), $A_e = 0.0368$ and $A_{n\phi} = 0.1584$. Now, $|A_{14}^1| > A_{n\phi} > A_e$; obviously the combination of λ_1 and λ_{14} can satisfy condition (19b). From Eq. (13a) we obtain $m_{14}' = 9540$ and $e_{14}'' = 0.726$; from Eq. (23), $\Delta m_{1,14} = 1833$, as shown in Table 3; from Eq. (24), $z = 1833 - 11379 + 9540 = -6$. Using Eq. (17), we obtain

$${}^sx = {}^1x = \text{Int}\left[\frac{0.31 - 0.726 - (-6)}{-0.161554087}\right] = \text{Int}(-34.54) = -35.$$

Now, from Eqs. (19a) we have $\Delta x_e = 0.12 < 0.5$ and $\Delta x_n + \Delta x_\phi = 0.4045 + 0.0874 = 0.4919 < 0.5$; hence ${}^1x = -35$ is correct. From Eq. (11), we have $m_s = m_1 = 11379 - 35 = 11344$; from Eq. (18), $m_u = m_{14} = 9540 - 35 - (-6) = 9511$. Using the obtained value of m_1 or m_{14} and knowledge of $\Delta m_{s,u}$, we can obtain the integral orders of other wavelengths as given in Table 3.

We know that the integral orders shown in Table 3 are correct in comparison with those in Table 1.

Finally, the effective optical thickness $n(\lambda)d$ can be obtained by Eq. (2) for each wavelength. The dispersion curve of $2n(\lambda)d$ in the range between $\lambda_1 = 5481.1314$ Å and $\lambda_{14} = 6521.3641$ Å can be obtained by a least-squares fit to the measured values $(m_i + e_i')\lambda_i$ and can be fitted well to a third-order polynomial as

$$2n(\lambda)d = 65683760.46 - 1317.838609\lambda + 1.641057598 \times 10^{-1}\lambda^2 - 7.363036946 \times 10^{-6}\lambda^3,$$

where λ is expressed in angstroms.

The Practical Method

The errors of the refractive index and the phase shift are considered separately in the practical method.

Consider the case in which $d' \pm \Delta d = 2.131 \pm 0.005$ mm, $C_\phi' \pm \Delta C_\phi = -0.60 \pm 0.05$, the measured e_i' and n_i' are as shown in Table 2, $\Delta n = 5 \times 10^{-6}$, $\Delta e = 0.01$, and the range of calibration is from 0.54811314 to 0.65213641 μm.

First, when we neglect the phase shift on reflection, we find the integral orders m''.

We choose the proper intervals of wavelengths according to condition (21) in the given range of calibration. We let $\lambda_s = \lambda_1 = 0.54811314$ μm; from expression (9) we have

Table 3. Calculated Results of the General Method

s	u	m_s'	A_u^s	m_u'	e_u''	z	$\Delta m_{s,u}$	$\Delta m_{1,u}$	m_u [a]
1	2	11379	−0.014660976	11212	0.232	0	167	167	11177
2	3	11212	−0.014441727	11050	0.759	−1	161	328	11016
3	4	11050	−0.014190795	10893	0.478	0	157	485	10859
4	5	10893	−0.013925084	10742	0.232	0	151	636	10708
5	6	10742	−0.013815791	10594	0.331	0	148	784	10560
6	7	10593	−0.013612135	10449	0.586	−1	143	927	10417
7	8	10449	−0.013416174	10308	0.825	−1	140	1067	10277
8	9	10309	−0.013216906	10173	0.013	0	136	1203	10141
9	10	10173	−0.013048088	10040	0.696	−1	132	1335	10009
10	11	10040	−0.012919540	9910	0.3573	0	130	1465	9879
11	12	9910	−0.012729757	9784	0.6384	−1	125	1590	9754
12	13	9784	−0.012585641	9660	0.8723	−1	123	1713	9631
13	14	9661	−0.012450697	9540	0.9912	−1	120	1833	9511

[a] $m_u = m_1 - \Delta m_{1,u} = 11344 - \Delta m_{1,u}$.

Table 4. Calculated Results of the Practical Method

s	u	m_s'	A_u^s	m_u'	e_u''	z	$\Delta m_{s,u}$	$\Delta m_{1,u}$	m_u [a]
1	2	11379	−0.014658675	11212	0.258	0	167	167	11177
2	3	11212	−0.014439427	11050	0.785	−1	161	328	11016
3	4	11050	−0.014188499	10893	0.504	0	157	485	10859
4	5	10893	−0.013922801	10742	0.255	0	151	636	10708
5	6	10742	−0.013813493	10594	0.356	0	148	784	10560
6	7	10593	−0.013609838	10449	0.610	−1	143	927	10417
7	8	10449	−0.013413879	10308	0.849	−1	140	1067	10277
8	9	10309	−0.013214614	10173	0.037	0	136	1203	10141
9	10	10173	−0.013045794	10040	0.720	−1	132	1335	10009
10	11	10040	−0.012917241	9910	0.3804	0	130	1465	9879
11	12	9910	−0.012727459	9784	0.6612	−1	125	1590	9754
12	13	9784	−0.012583342	9660	0.8948	−1	123	1713	9631
13	14	9661	−0.012448391	9541	0.0134	0	120	1833	9511

[a] $m_u = m_1 - \Delta m_{1,u} = 11344 - \Delta m_{1,u}$.

$$n'(\lambda_s)d' = n_s'd' = n_1'd' = 3111.230166\ \mu\text{m}$$

and

$$\Delta[n'(\lambda_s)d'] = \Delta(n_s'd') = \Delta nd' + n_s'\Delta d = \Delta(n_1'd') = 7.310585\ \mu\text{m};$$

from relation (12),

$$|^s x_m| = \frac{4\Delta(n_s'd')}{\lambda_s} = |^1 x_m| = 53.35;$$

from Eq. (10),

$$m_s' = \text{Int.P}\left\{\frac{2[n_s'd' + \Delta(n_s'd')]}{\lambda_s}\right\} = m_1' = 11379.$$

If $\lambda_u = \lambda_2 = 0.55614231\ \mu$m, then from Eq. (15) we have

$$A_u^s = \frac{n_u'\lambda_s}{n_s'\lambda_u} - 1 = A_2^1 = -0.014658675;$$

from Eqs. (8a) and (8b),

$$\delta = \delta_e + \delta_n + \delta_\phi = 0.0199 + 0.0766 + 0.0046 = 0.1011;$$

from relation (21), $Am = 0.0149$. Now $|A_2^1 \cdot {}^1x_m| = 0.78 < 1$, $|A_2^1| = 0.014658675 < Am$; conditions (16) and (21) have been satisfied. From Eq. (13a) we have $m_u' = m_2' = 11212$ and $e_u'' = e_2'' = 0.258$. According to the selection rules of z, now $A_2^1 \cdot {}^1x_m > 0$, $e_2' > e_2''$; thus we select $z = 0$, and, from Eq. (20), $\Delta m_{1,2} = 167$. In a similar manner we can obtain other $\Delta m_{s,u}$ as shown in Table 4. The integral-order differences between the nonadjacent wavelengths can be obtained by using Eq. (23).

We then choose the proper combination of wavelengths according to the requirements that $\Delta x_e < 0.5$ and $\Delta x_n < 0.5$. If $\lambda_u = \lambda_{12} = 0.63610744\ \mu$m, then from Eq. (15) we have $A_{12}^1 = -0.140139609$; from Eq. (19b) we have $A_e = 0.0372$ and

$$A_n = \frac{8d'\Delta n}{\lambda_u} = 0.1340.$$

Now $|A_{12}^1| > A_n > A_e$; thus the combination of λ_1 and λ_{12} satisfies the requirements of $\Delta x_e < 0.5$ and $\Delta x_n < 0.5$. From Eq. (13a) we have $m_{12}' = 9784$ and $e_{12}'' = 0.403$; from Eq. (23), $\Delta m_{1,12} = 1590$ as shown in Table 4; from Eq. (24), $z = 1590 - 11379 + 9784 = -5$. From Eq. (17) we obtain

$${}^s x = {}^1 x = \text{Int}\left(\frac{0.01 - 0.403 + 5}{-0.140139609}\right) = \text{Int}(-32.9) = -33.$$

Now, from Eqs. (19a) we have $\Delta x_e = 0.14 < 0.5$ and $\Delta x_n = 0.48 < 0.5$; hence ${}^1x = -33$ is correct for the case in which we neglect the phase shift on reflection. From Eq. (11) we have $m_s'' = m_1'' = 11379 - 33 = 11346$; from Eq. (18), $m_u'' = m_{12}'' = 9784 - 33 + 5 = 9756$.

Second, we find m_ϕ with the known C_ϕ'. From Eq. (25) we obtain

$$m_\phi = \text{Int}(2n_u'C_\phi') = \text{Int}(2n_{12}'C_\phi') = \text{Int}[2 \times 1.456924(-0.60)] = \text{Int}(-1.7) = -2.$$

From relation (25a) we know that $\Delta C_\phi < 0.08$ is required; now $\Delta C_\phi = 0.05 < 0.08$, and hence $m_\phi = -2$ is correct.

Finally, from Eq. (26) we obtain $m_s = m_1 = 11346 - 2 = 11344$ and $m_u = m_{12} = 9756 - 2 = 9754$. By using the obtained m_1 or m_{12} value and knowledge of $\Delta m_{s,u}$, we can obtain the integral orders of other wavelengths as shown in Table 4. Obviously, these results are correct as well.

The Correctional Method

When the requirements that $\Delta x_n < 0.5$ and/or $\Delta x_e < 0.5$ are not satisfied, i.e., when the correct value of ${}^sx$ cannot be computed unambiguously from Eq. (17), the correctional method is useful. In this method we first find an approximate integral order m''', and then by using our knowledge of C_ϕ, we find the correct Δm.

Consider the case in which $d' \pm \Delta d = 2.131 \pm 0.005$ mm, $C_\phi' \pm \Delta C_\phi = -0.55 \pm 0.11$, the usable spectral range is from $\lambda_4 = 0.57216534\ \mu$m to $\lambda_{11} = 0.62811012\ \mu$m, the measured values of e_i' and n_i' are shown in Table 2 from λ_4 to λ_{11}, $\Delta n = 5 \times 10^{-6}$, and $\Delta e = 0.01$. Under these conditions, if we use the practical method, we find $m_s' = m_4' = 10893$, the largest $A_u^s = A_{11}^4 = -0.090239691$, $m_u' = m_{11}' = 9910$, $e_u'' = e_{11}'' = 0.865$, $\Delta m_{4,11} = 980$, $z = -3$, ${}^4x = -33$, and $\Delta x_n = 0.75 > 0.5$. Therefore we can obtain only the approximate values $m_4''' = 10860$ and $m_{11}''' = 9880$ and the other m''' values shown in Table 5.

By using the initial n_i' and Eq. (31), we can obtain $d'''(\lambda_i)$ as shown in Table 5. After the obtained $d'''(\lambda_i)$ is fitted to a

Table 5. Obtained Approximate m_i''' and $d'''(\lambda_i)$ with the Correctional Method

i	λ_i (Å)	m_i'''	$d'''(\lambda_i)$ (Å)
4	5721.6534	10860	21295833.41
5	5801.2346	10709	21295902.12
6	5881.3471	10561	21295859.89
7	5961.3796	10418	21295786.79
8	6041.3215	10278	21295799.07
9	6121.1367	10142	21295785.14
10	6201.0134	10010	21295670.64
11	6281.1012	9880	21295740.63

Table 6. Linear Fitting of $d'''(\lambda_i)^a$

i	λ_i (Å)	P_i	P_i^2	$d'''(\lambda_i)P_i \times 10^{-5}$
4	5721.6534	−279.620037	78187.36509	−59547.41726
5	5801.2346	−200.038837	40015.5363	−42600.07492
6	5881.3471	−119.926337	14382.3263	−25539.34469
7	5961.3796	−39.893837	1591.51823	−8495.706469
8	6041.3215	40.048063	1603.84735	8528.555027
9	6121.1367	119.863263	14367.20181	25525.82295
10	6201.0134	199.739963	39896.05281	42535.96465
11	6281.1012	279.827763	78303.57694	59591.39461
Total	48010.1875		268347.42483	−0.806102

[a] $P_i = \lambda_i - a_1$, $a_1 = (\sum_{i=4}^{11}\lambda_i)/(\sum_{i=4}^{11}1) = 48010.1875/8 = 6001.273437$, $d_1 = [\sum_{i=4}^{11} d'''(\lambda_i)P_i]/(\sum_{i=4}^{11} P_i^2) = (-0.806102 \times 10^5)/268347.42483 = -0.300394908$.

first-order polynomial, from Eq. (32) we obtain $C_\phi''' = d_1 = -0.300394908$ as shown in Table 6. By using Eq. (29) we obtain

$$\Delta m = \mathrm{Int}[2n(C_\phi' - C_\phi''')] = \mathrm{Int}[2 \times 1.458(-0.55 + 0.300)] = \mathrm{Int}(-0.729) = -1,$$

where n may take its mean value. If $\Delta C_\phi''' = 0.05$, we know from relation (30) that the condition that $\Delta C_\phi < 0.12$ is required; now $\Delta C_\phi = 0.11 < 0.12$, and hence $\Delta m = -1$ is correct.

Finally, by using Eq. (27) we can obtain the correct value $m_4 = 10860 - 1 = 10859$ and other integral orders as shown in Table 3 from λ_4 to λ_{11}. Obviously, these results are correct as well.

DISCUSSION

It should be pointed out that the integral-order error in the calibration has been contained within the calibrated optical thickness because it is the product of the order of interference and the standard wavelengths in the present method of calibration. If the errors of the fractional orders could be neglected, the measured value of the wavelength would not be affected by the integral-order error in the calibration; that is,

$$\lambda = \frac{2nd'''(\lambda)}{m''' + e} = \frac{2nd(\lambda)}{m + e}.$$

This equation can easily be obtained from Eqs. (27) and (28). Even though the fractional-order errors are taken into consideration, the measured wavelength is

$$\lambda = \frac{2nd'''(\lambda)}{m''' + e},$$

and its uncertainty is

$$\Delta\lambda = \frac{(\Delta e_d + \Delta e)}{m''' + e} = \frac{(\Delta e_d + \Delta e)}{m - \Delta m + e}, \qquad (33)$$

where Δe_d and Δe are the errors of the fractional orders in the calibration of the effective optical thickness and in the measurement of the wavelength. It is obvious that the variation of the uncertainty of the wavelength is small as long as $\Delta m \ll m$. Specifically, when the contribution of the phase shift on reflection to the integral order is less than the permissible error of the integral order for the precise measurement of the wavelength, the phase shift may be neglected. In fact, even though we do not have detailed knowledge of the phase shift of reflection on the films, if we start with $|\phi| < \pi$ in the reflecting range for a dielectric multilayer, we have undoubtedly

$$|C_\phi| < \frac{\lambda_b}{2n|\lambda_b - \lambda_0|}, \qquad (34)$$

where λ_b is the wavelength of the edge of the high-reflecting range. Hence from Eq. (25) we obtain

$$|m_\phi| < \frac{\lambda_b}{|\lambda_b - \lambda_0|}.$$

For example, for a general dielectric multilayer film with $\lambda_0 = 0.6\ \mu$m and a linear phase-shift range from 0.55 to 0.65 μm, we undoubtedly have $|m_\phi| < 13$. As can readily be seen from Eq. (33), under the condition that $d \gg \lambda$, the phase shift of reflection can essentially be neglected in the present method of calibration for a precise measurement of the wavelength. However, the integral-order difference between adjacent wavelengths must be correct; i.e., condition (21) must be satisfied for the measurement of the wavelength. If detailed knowledge of the phase shift on the reflecting films is not available, we may estimate the value of δ_ϕ under condition (34). The interval of adjacent wavelengths corresponding to such a value of δ_ϕ will ensure the correctness of $\Delta m_{s,u}$ if the phase shift is neglected.

SUMMARY

A general method of calculating the integral-order number of interference is developed, completely without a trial method. The developed method of excess fractions is extended to the calibration of the effective optical thickness of a Fabry–Perot étalon with the dispersion of the refractive index and the phase shift on reflection. Several useful methods of calibration are provided and are illustrated by examples. The condition under which the phase shift can be neglected is given for calculating the correct integral-order difference between wavelengths. It is pointed out that exact knowledge of the integral order is not even necessary and that the phase shift can thus essentially be neglected for a precise measurement of wavelength as long as the effective optical thickness is calibrated as the product of the fringe order of interference and the standard wavelength and as long as the proper intervals of wavelengths are chosen. A possible method for precise measurement of the dispersive phase shift is given.

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