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Construction of General Unitary Transformation for Conditional Quantum Dynamics and Realization of Quantum Controlled Gates *

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By factorizing a general one-bit transformation matrix for one-bit gates in quantum computer, a general unitary transformation is constructed, which can serve as (controlled)^m gate for the conditional quantum dynamics. When the quantized single-mode electromagnetic field containing n or no photons acts as the controlled bit, the quantum controlled-NOT gates and square root of the controlled-NOT gates in cavity-quantum electrodynamics are realized .

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Recently, there is considerable interest in quantum computation.[1−6] The combination of quantum mechanics and computer science makes great developments in information theory. The quantum extension of conventional Boolean bits is called qubits (two-state systems). The examples include the spin-half particle[4] with spin up corresponding to 1 and spin down to 0,[4] the polarization of a photon with clockwise polarization corresponds to 1 and counterclockwise to 0,[5,6] and two-state atom with excited state corresponds to 1 and ground state to 0.[5] The computation proceeds by manipulating these qubits using quantum logic gates which are input-output devices (unitary transformations). The quantum logic gates are combined together to form a quantum network and the quantum network forms a quantum computer.

Duetsch[7] proposed a universal three-bit logic gate and Divencenzo[8] showed later that this three-bit gate can be implemented by an arrangement of two-bit gates. Sleator *et al.*[5] showed then that a two-bit gate is universal and made an interesting experiment aiming at constructing quantum gate prototypes with cavity-QED (quantum electrodynamics) techniques. They only proposed the quantum (controlled)^1 gates, the most general (controlled)^m gates were not considered.

The most important two-bit gates are represented by a 4×4 unitary transformation matrix U in the computational basis $\{|a\rangle_1|b\rangle_2\}$, the direct product of the two qubits, $a, b \in \{0,1\}$,[2]

$$U = \begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix} \tag{1}$$

with

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} \cos\phi & \exp(-\mathrm{i}\,\theta)\sin\phi \\ -\exp(\mathrm{i}\,\theta)\sin\phi & \cos\phi \end{pmatrix}, \tag{2}$$

where ϕ and θ are real values.

The first bit a is called the control bit and the second bit b the target bit. Under the transformation U, the control bit is never changed. But it decides whether the target bit undergoes a unitary transformation. When the control bit is in state $|0\rangle_1$, the target bit is invariant. When the control bit is in state $|1\rangle_1$, the target bit experiences a unitary transformation described by V. This is the so-called "conditional quantum dynamics".

For $\theta = -\pi/2$ and $\phi = \pi/2$, the transformation U reduces to

$$U' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix}. \tag{3}$$

Besides a common phase factor, this is the quantum controlled-NOT gate which has many applications such as quantum state swapping and quantum state teleportation.[4] In order to construct a general unitary transformation for the conditional quantum dynamics, we try to factorize the transformation V.

The one-bit transformation matrix V can be expressed in terms of Pauli matrixes $\boldsymbol{\sigma}$ as

$$V = \cos\phi - \mathrm{i}\,\sin\phi[\sin(\theta)\sigma_x - \cos(\theta)\sigma_y]. \tag{4}$$

By using the relations

$$\exp(\frac{\mathrm{i}\,\lambda}{2}\sigma_z)\sigma_x\exp(-\frac{\mathrm{i}\,\lambda}{2}\sigma_z) = \cos(\lambda)\sigma_x - \sin(\lambda)\sigma_y,$$
$$\exp(-\mathrm{i}\,\lambda\sigma_x) = \cos(\lambda)I - \mathrm{i}\,\sin(\lambda)\sigma_x, \tag{5}$$

the factorization form of V is obtained:

$$V = \exp\left[\mathrm{i}\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\sigma_z\right]\exp(-\mathrm{i}\,\phi\sigma_x) \cdot \exp\left[-\mathrm{i}\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\sigma_z\right]. \tag{6}$$

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From this form, we construct the general unitary transformation formally as

$$W = \exp\left[\mathrm{i}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\sigma_z\right]\exp(-\mathrm{i}\,\phi\hat{F}\sigma_x) \cdot\exp\left[-\mathrm{i}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\sigma_z\right], \quad (7)$$

where $\hat{F}$ is an operator of additional subsystem. The operator commutes with all the Pauli matrixes and couples with σ_x. We assume that it has two special eigenvalues among its set of eigenvalues:

$$\hat{F}|0\rangle_1 = 0|0\rangle_1 = 0,$$
$$\hat{F}|1\rangle_1 = 1|1\rangle_1 = |1\rangle_1. \quad (8)$$

The realization of the operator $\hat{F}$ in the real physical system will be seen below.

Let the subsystem represented by operator $\hat{F}$ be the control subsystem and another one represented by Pauli operators be the target subsystem. It is easily found that the matrix U is the representation of unitary transformation W in the computational basis. That is to say, the target bit undergoes a unitary transformation V if and only if the control bit is 1. Thus, the conditional quantum dynamics is realized by introducing an additional operator $\hat{F}$.

The unitary transformation W can be generalized to a more general case

$$W' = \exp\left[\mathrm{i}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\sigma_z\right]\exp\left[-\mathrm{i}\,\phi\left(\prod_{n=1}^{m}\hat{F}_n\right)\sigma_x\right] \cdot\exp\left[-\mathrm{i}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\sigma_z\right], \quad (9)$$

where m is a positive integer and the operator $\hat{F}_n$ has the same property as $\hat{F}$. The above gate is the so-called (controlled)^m gate which flips the target gate if and only if the first m bits are 1.

Next, we proceed to the problem of how to realize a two-bit gate in a realistic physical system. Barenco *et al.*[4] have proposed a method to construct a quantum controlled-NOT gate in cavity-QED and the field state is either in the Fock state $|1\rangle$ or the vacuum state $|0\rangle$. In what follows, we will extend their methods to realize two-bit quantum gates in which the field state may be in the Fock state $|n\rangle$ (n is a positive integer).

Let the control qubit be the quantized field which may contain n photons and the target bit be the atom with two circular Rydberg states $|b\rangle_2 (b \in \{0,1\})$, $|1\rangle_2$ denotes the excited state and $|0\rangle_2$ the ground state. The atom crosses through three regions sequentially: Ramsey zone R, cavity C, (off-resonant interaction), and another Ramsey zone R^{-1}. The Ramsey zone consists of a classical rf field and produces θ rotation of Block vector of an atom along y-axis in spin space. In cavity C, the atom interacts with single-mode quantized cavity field. Let the detuning between the atomic transition frequency and the cavity field frequency be much larger than the coupling constant, the effective Hamiltonian can be written as [10-12]

$$H = \phi\sigma_z a^+ a, \quad (10)$$

where $\phi = g^2/\Delta$, g is the coupling constant and Δ is the detuning; a^+ and a are the creation and annihilation operators of cavity field, respectively. When the atom passes through the three regions, the resulting transformation is

$$\overline{W}(\theta,\phi) = \exp(\mathrm{i}\,\theta\sigma_y)\exp(-\mathrm{i}\,\phi a^+ a\sigma_z)\exp(-\mathrm{i}\,\theta\sigma_y). \quad (11)$$

It should be noted that the transformation $\sigma_x \to \sigma_y, \sigma_y \to \sigma_z$, and $\sigma_z \to \sigma_x$ can be realized by the appropriate choice of another basis. After the transformation, Eq.(11) changes into

$$\overline{W}'(\theta,\phi) = \exp(\mathrm{i}\,\theta\sigma_z)\exp(-\mathrm{i}\,\phi a^+ a\sigma_x)\exp(-\mathrm{i}\,\theta\sigma_z). \quad (12)$$

By comparing Eqs. (7) with (12), the number operator a^+a corresponds to the operator $\hat{F}$. The operator a^+a is a physical realization of formally introduced operator $\hat{F}$ in the cavity-QED.

The transformation $\overline{W}(\theta,\phi)$, Eq. (11), can be expressed in the following form:

$$\overline{W}(\theta,\phi) = \cos(a^+a\phi) - \mathrm{i}\,\sin(a^+a\phi)[\cos(\theta)\sigma_z - \sin(\theta)\sigma_x]. \quad (13)$$

By assuming that the cavity field is in Fock state $|n\rangle_1$ and the atom in state $|b\rangle_2, b \in \{0,1\}$, the final state under the transformation is

$$[\cos(n\phi) - \mathrm{i}\,(-1)^{b+1}\sin(n\phi)\cos\theta]\,|n\rangle_1|b\rangle_2 + \mathrm{i}\,\sin(n\phi)\sin\theta|n\rangle_1|1-b\rangle_2. \quad (14)$$

When the field is in the state $|0\rangle_1$, the atomic state does not change. When the field contains $n (n \neq 0)$ photons, the atom state will change to

$$[\cos(n\phi) - \mathrm{i}\,(-1)^{b+1}\sin(n\phi)\cos\theta]|b\rangle_2 + \mathrm{i}\,\sin(n\phi)\sin\theta|1-b\rangle_2. \quad (15)$$

Let $n\phi = \theta = \pi/2$, we can write the transformation $\overline{W}(\pi/2, \pi/(2n))$ explicitly as

$$|0\rangle_1|0\rangle_2 \longrightarrow |0\rangle_1|0\rangle_2,$$
$$|0\rangle_1|1\rangle_2 \longrightarrow |0\rangle_1|1\rangle_2,$$
$$|n\rangle_1|0\rangle_2 \longrightarrow \mathrm{i}\,|n\rangle_1|1\rangle_2,$$
$$|n\rangle_1|1\rangle_2 \longrightarrow \mathrm{i}\,|n\rangle_1|0\rangle_2. \quad (16)$$

If we express the transformation in the computational basis $\{|0\rangle_1|0\rangle_2, |0\rangle_1|1\rangle_2, |n\rangle_1|0\rangle_2, |n\rangle_1|1\rangle_2\}$ in matrix form, it is just the matrix in Eq. (3). The quantum controlled-NOT gate is realized. Let $n\phi = \pi/4, \theta = \pi/2$ in Eq.(14), the unitary transformation can be written in the computational basis as

$$\overline{W}\left(\frac{\pi}{2}, \frac{\pi}{(4n)}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & \mathrm{i}/\sqrt{2} \\ 0 & 0 & \mathrm{i}/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}. \quad (17)$$

We note that $\overline{W}^2(\pi/2, \pi/(4n)) = \overline{W}(\pi/2, \pi/(2n))$, so the gate $\overline{W}(\pi/2, \pi/(4n))$ can be considered as a square root of quantum controlled-NOT gate. It should be pointed out that the square root of quantum controlled-NOT gate is an important component for constructing three-bit Toffoli gate.[5]

In summary, we have factorized the one-bit transformation matrix and constructed the general unitary transformation which is considered as (controlled)^m gate for conditional quantum dynamics. By using the cavity-QED technique, when the cavity field contains either n photons or no photons, we have realized not only the quantum controlled-NOT gate but also the square root of the controlled-NOT gate.

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