Redistribution of output weighting coefficients for complex multiplexed phase-diffractive elements

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Abstract: The formation of multiplexed phase-only holograms with more weighted phase functions creates spurious cross terms and nonlinear scaling. We extend previously reported work [Appl. Opt. **25**, 3767 (1986)] by proposing a normal method to analyze multiplexed holograms mathematically. We show that the output of holograms with any number weighted phase function can be written as a new linear combination for the original phase function with new weights. The relationship between the original weights and the new weights is developed for real-time optimization of hologram performance. We focus on the analysis of two and three multiplexed holograms to demonstrate the effectiveness of this approach.

OCIS codes: (090.4220) Multiplex holography; (230.4110) Modulators

References and links

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1. Introduction

There are several applications including optical pattern recognition [1], optical interconnections [2,3], three-dimensional display, and an absolute interferometric test of aspherics [4,5], in which several phase functions with different weights are multiplexed into a single phase-only or a binary phase-only hologram. In many cases, the multiplexed functions

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are added and coded onto a spatial light modulator (SLM) such as a magneto-optic spatial light modulator, twisted nematic liquid crystals, and transmissive matrix addressed ferroelectric liquid crystals. Nonlinearity occurs in these processes because each pixel of the SLM must encode the sum of these functions. Research has shown that spurious cross terms are formed and the weights of the output have a nonlinear relation with the weights of the encoded linear combination [6], which is a fundamentally different result compared with the desired output. This problem forces one to use a variety of complicated numerical techniques to compensate for the nonlinearities [7–9]. The conventional way to tackle these kinds of problem is to use various computer algorithms such as simulated annealing or iterative gradient approaches. However, these approaches are computer intensive and not intuitive. We propose a new and simple way to analyze the hologram constructed by adding two weighted phase functions [10]. However this research is limited to the analysis of two multiplexed phase-diffractive elements. The particular technique of this method cannot be extended to multiplexed holograms with more than two phase functions. To our knowledge, no previous research mathematically deduced the relationship between the encoded weights and the output weights with a normal method that is suitable for holograms with *N*-weighted phase functions.

We extend previously reported results [8] by proposing a normal method to analyze mathematically multiplexed holograms with *N*-weighted phase functions. We show that the output can be written as a new linear combination for the original phase functions with new weights. The relationship between the original weights and the new weights is developed for real-time optimization of hologram performance. We use a fast and efficient way to fit polynomials to obtain the weights for multiplexed holograms that correspond to the desired output and efficiency. In particular we analyze a trifocal hologram with our method. In addition, a bifocal hologram can be used in lieu of a trifocal hologram under some conditions, and the results obtained with the trifocal hologram are identical with those reported in Ref. 8, which corroborates our method. It is believed that holograms with less than *N* phase functions can be used without the need for analysis when we analyze a hologram with *N*-weighted phase functions.

2. Theory

We consider a linear combination of *N* phase functions $\exp(i\phi_n)$ (ϕ_n, where n = 1,2…*N* are two-dimensional functions) with real weights A_n we assume that $\sum_{i=1}^{n} A_i^2 = 1$, which defines a new function $M\exp(ia)$, where *M* is the amplitude and a is the phase:

$$M\exp(ia) = \sum_{n=1}^{N} A_n \exp(i\phi_n) \tag{1}$$

We need to perform some simple transformations in Eq. (1) to obtain a useful expression:

$$\exp(ia) = \sum_{n=1}^{N} \frac{A_n}{M}\exp(i\phi_n) \tag{2}$$

where

$$\begin{aligned} M = [&A_1^2 + A_2^2 + \cdots + A_N^2 + 2A_1A_2\cos(\phi_1-\phi_2) + 2A_1A_3\cos(\phi_1-\phi_3) + \cdots + 2A_1A_N\cos(\phi_1-\phi_N) \\ &+ 2A_2A_3\cos((\phi_1-\phi_3)-(\phi_1-\phi_2)) + \cdots + 2A_2A_N\cos((\phi_1-\phi_N)-(\phi_1-\phi_2)) \\ &+ \cdots\cdots \\ &+ 2A_{N-1}A_N\cos((\phi_1-\phi_N)-(\phi_1-\phi_{N-1}))]^{1/2} \end{aligned} \tag{3}$$

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For $M > 0$, a key point in our development is that $1/M$ can be considered a periodic function of $(\phi_1 - \phi_2)$, $(\phi_1 - \phi_3) \ldots (\phi_1 - \phi_N)$ with period 2π. This allows a Fourier series expansion to be performed:

$$M^{'}(\beta_1\beta_2 \cdots \beta_{N-1}) = \frac{1}{M} = \sum_{m_1} \cdots \sum_{m_{N-1}} a_{m_1 m_2 \cdots m_{N-1}} \exp(im_1\beta_1 + im_2\beta_2 + \cdots + im_{N-1}\beta_{N-1}) \tag{4}$$
$$\beta_1 = \phi_1 - \phi_2 \cdots \beta_{N-1} = \phi_1 - \phi_{N-1}$$

Here m_1, $m_2 \ldots m_{n-1}$ are integral numbers, and weights $a_{m_1 m_2 \cdots m_{N-1}}$ can be written as

$$a_{m_1 m_2 \cdots m_{N-1}} = \frac{1}{(2\pi)^{N-1}} \int_0^{2\pi} \cdots \int_0^{2\pi} M^{'} \exp(-im_1\beta_1 - \cdots - im_{N-1}\beta_{N-1}) d\beta_1 \cdots d\beta_{N-1} \tag{5}$$

Here two main properties of weights $a_{m_1 m_2 \cdots m_{N-1}}$ can be obtained. First, weights $a_{m_1 m_2 \cdots m_{N-1}}$ are real, because, from Eq. (3), $M^{'}(\beta_1, \beta_2 \cdots \beta_{N-1}) = M^{'}(-\beta_1, -\beta_2 \cdots -\beta_{N-1})$. Second, from Eqs. (3) and (5) $a_{m_1 m_2 \cdots m_{N-1}}$ depends only on A_n and n and is completely independent of ϕ_n. Inserting Eq. (4) into Eq. (2) and using the definition of $\beta_1 \cdots \beta_{N-1}$ given in Eq. (4) finally yield

$$\exp(ia) = \sum_{m_1} \cdots \sum_{m_{N-1}} a_{m_1 \cdots m_{N-1}} \begin{Bmatrix} A_1 \exp[i(m_1 + \cdots + m_{N-1} + 1)\phi_1 - im_1\phi_2 - \cdots - im_{N-1}\phi_N] \\ + A_2 \exp[i(m_1 + \cdots + m_{N-1})\phi_1 - i(m_1 - 1)\phi_2 - \cdots - im_{N-1}\phi_N] \\ + \cdots \\ + A_N \exp[i(m_1 + \cdots + m_{N-1})\phi_1 - im_1\phi_2 - \cdots - i(m_{N-1} - 1)\phi_N] \end{Bmatrix} \tag{6}$$

This new function leads to two consequences. First, the phase-only diffractive element corresponding to N-multiplexed phase functions can be written as a new linear combination of the original functions affected by the new weights a_n. Second, the phase-only operation can introduce spurious cross terms. Since the new weights of the output depend only on the original weight of the input, one can perform an excellent polynomial fit on their dependence.

3. Validation

If $N = 3$, according to Eq. (6) the phase-only diffractive element that corresponds to three multiplexed phase functions can be written as

$$\exp(ia) = \cdots + (a_{00}A_1 + a_{10}A_2 + a_{01}A_3)\exp(i\phi_1) + (a_{00}A_2 + a_{-10}A_1 + a_{-11}A_3)\exp(i\phi_2) \\ + (a_{00}A_3 + a_{0-1}A_1 + a_{1-1}A_2)\exp(i\phi_3) + \cdots \tag{7}$$

It is clear that the output can be written as a new linear combination of the original functions and spurious cross terms introduced. The new weights depend only on the original weights. We define the output ratios as $x_1 = (a_2/a_1)^2$ and $x_2 = (a_3/a_1)^2$, the input ratios as $y_1 = (A_2/A_1)^2$ and $y_2 = (A_3/A_1)^2$. Then the relationship between the input ratios and the output ratios shown in Fig. 1 can be obtained quickly and efficiently by use of a fast Fourier transform (FFT) operation. The diffraction efficiency, defined as $a_1^2 + a_2^2 + a_3^2$, is plotted in Fig. 2 as a function of the desired output ratios.

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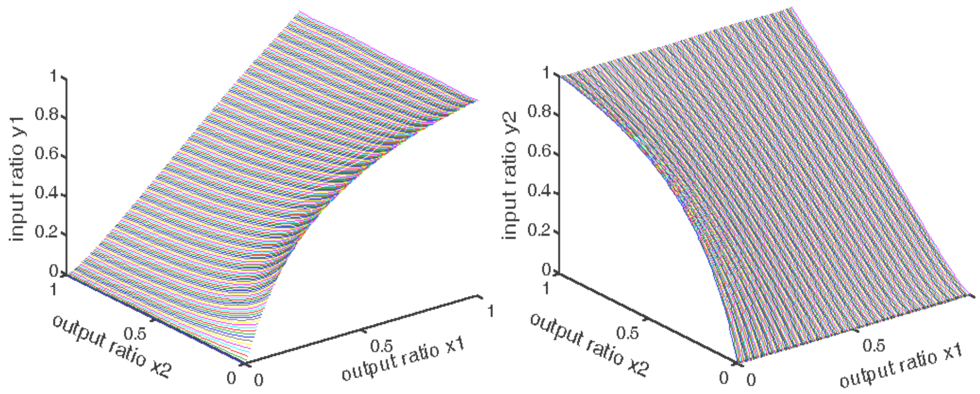


(a) (b)

Fig. 1. Input ratio versus the desired output ratio of the three multiplexed holograms.

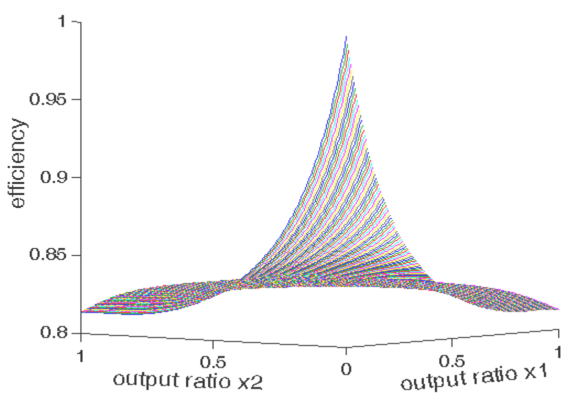


Fig. 2. Diffraction efficiency versus output ratio of the three multiplexed holograms.

Note that the relationship between the input ratios and the output ratios can be written as polynomials:

$$y_2 = \sum_{i=0}^{k}\sum_{j=0}^{i} m_{ij} x_1^j x_2^{i-j} \ , \ y_1 = \sum_{i=0}^{k}\sum_{j=0}^{i} n_{ij} x_1^j x_2^{i-j} \tag{8}$$

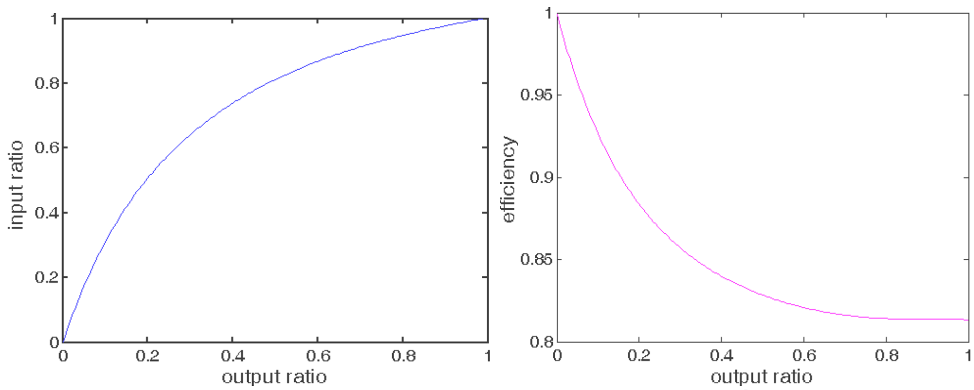
where k is the degree of these polynomials. The polynomial coefficients can be determined by performing a least-squares fit of the polynomial on the dependence of input and output ratios. Here we chose $k = 7$ and the polynomials are

$$y_1= 0.00005096072781+3.94783x_1-0.00873x_2-10.19480x_1^2+\ldots\ldots+ 11.96649x_1^2x_2^5+ 4.82817x_1x_2^6-4.11001x_2^7 \tag{9}$$

$$y_2= 0.00005096072782-0.00873x_1+3.94783x_2+0.44625x_1^2+\ldots\ldots+ 8.09256x_1^2x_2^5+ 6.93143\, x_1x_2^6+ 2.49845x_2^7 \tag{10}$$

The maximum absolute error of y_1 and y_2 is less than 0.002. We could achieve higher accuracy by increasing the degree of the polynomial and the sampling number. Now we can control the weights of the output in real time by adjusting the weights of the input.

It should be noted that the mathematical result of the two multiplexed holograms can be obtained from that of the three multiplexed holograms with $y_1 = 0$ (results in $x_1 = 0$) or $y_2 = 0$ (results in $x_2 = 0$). So the relationship between the input ratio and the output ratio of the two multiplexed holograms shown in Fig. 3 can be obtained quickly from Fig. 1 [with $x_2 = 0$ in Fig. 1(a) or with $x_1 = 0$ in Fig. 1(b)]. The diffraction efficiency shown in Fig. 3(b) can be obtained from Fig. 2 with $x_1 = 0$ or $x_2 = 0$. It is clear that the numerical results are the same as those reported in Ref. [8].



(a) (b)

Fig. 3 (a) Input ratio and (b) diffraction efficiency versus the desired output ratio of two multiplexed holograms.

4. Simulation

As an example, we created a binary phase-only hologram with $N = 3$ by quantifying the two-dimensional phase function with our computer program. Many possibilities can be considered. We consider the particular case of three Fresnel lens with the same focal length f, one equivalent to a center lens and each of the other two shifted from the origin by an amount a and b in the x direction. If $x_1 = 0.5$ and $x_2 = 0.6$, with Eqs. (9) and (10) we obtain $y_1 = 0.5383$ and $y_2 = 0.6564$. This implies that $A_1 = 0.7623$, $A_2 = 0.4038$, and $A_3 = 0.5004$. In such a case $\phi_1 = (k/2f)(x^2+y^2)$, $\phi_2 = (k/2f)\left[(x-a)^2+y^2\right]$, and $\phi_3 = (k/2f)\left[(x+b)^2+y^2\right]$, where k is the wave number. Substituting the phase functions above into Eq. (6), we can write the resulting trifocal lens as

$$\exp i(a) = \sum_{m_1}\sum_{m_2} a_{m_1 m_2} \left\{ \begin{aligned} & A_1 \exp\{\frac{ik}{2f}[[x+(m_1 a - m_2 b)]^2 + y^2]\} \exp[\frac{ik}{2f}(-(m_1 a - m_2 b)^2 - m_1 a^2 - m_2 b^2)] \\ & + A_2 \exp\{\frac{ik}{2f}[[x+(m_1 a - m_2 b - a)]^2 + y^2]\} \exp[\frac{ik}{2f}(-(m_1 a - m_2 b - a)^2 - m_1 a^2 - m_2 b^2 + a^2)] \\ & + A_3 \exp\{\frac{ik}{2f}[[x+(m_1 a - m_2 b + b)]^2 + y^2]\} \exp[\frac{ik}{2f}(-(m_1 a - m_2 b + b)^2 - m_1 a^2 - m_2 b^2 + b^2)] \end{aligned} \right\} \tag{11}$$

Here, substituting some pairs of numbers m_1 and m_2, such as $m_1 = 0$, $m_2 = 0$; $m_1 = 1$, $m_2 = 0$; $m_1 = 0$, $m_2 = 1$; $m_1 = -1$, $m_2 = 0$; $m_1 = 0$, $m_2 = -1$; $m_1 = -1$, $m_2 = 1$; $m_1 = 1$, $m_2 = -1$, into Eq. (11), we obtain the desired output orders that posit at $x = -b$, $x = 0$ and $x = +a$ as shown in Eq. (7). At the same time, we also obtain the spurious adjustments that appear at positions $x = (m_1 a - m_2 b)$, $x = (m_1 a - m_2 b - a)$ or $x = (m_1 a - m_2 b + b)$. Note that some spurious orders could coincide with the desired Fresnel lenses but with different phase functions. This could create a difference between the real weight of the output and the calculated weights. However, the problem can be compensated by taking these unwanted orders into account, when we calculate the relationship between the input ratios and the output ratios according to Eqs. (5) and (7).

A binary representation of a trifocal hologram is shown in Fig. 4(a) with a = 450 µm, b = 525 µm, and f = 1.138 m for a wavelength of 632.8 nm (He–Ne laser). The reconstruction of the hologram calculated by a Fourier transform of the hologram is shown in Fig. 4(b), in which the results are in agreement with the expected ratios. In addition, the locations of the peaks are at x = -525 µm, x= 0, and x = 450 µm, in agreement with Eq. (11). Some spurious

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orders appear at positions $x = (m_1a\text{-}m_2b)$, $x = (m_1a\text{-}m_2b - a)$, or $x = (m_1a\text{-}m_2b + b)$.

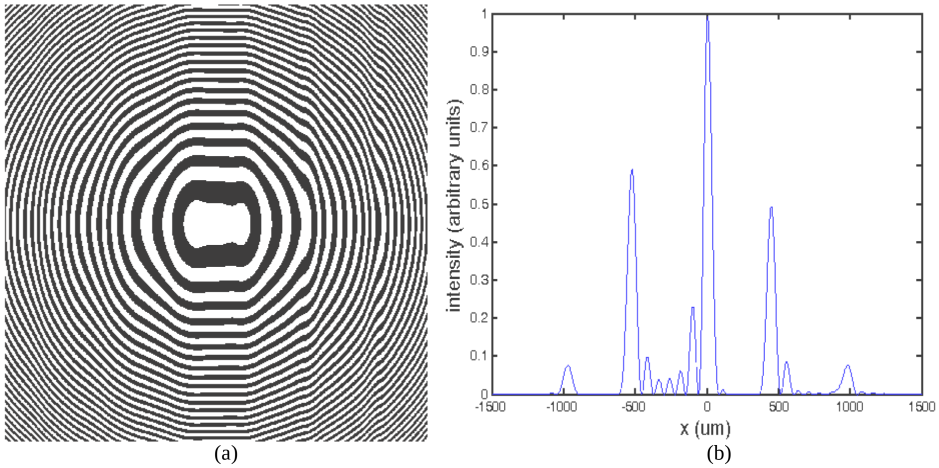

Fig. 4. (a) Binary representation of a trifocal lens desired for $x_1 = 0.5$ and $x_2 = 0.6$ and (b) reconstruction.

5. Conclusion

In conclusion, the phase-diffractive element constructed from a linear combination of *N*-weighted phase functions has been studied mathematically. We have shown that the final diffractive element contains a linear combination of the original phase functions affected by new weights and that some undesirable additional terms were created. These new weights depend only on the original weights of the input. Polynomial fitting is used to control the performance of the holograms quickly and efficiently. As an example, we studied a phase-diffractive element constructed from a linear combination of three weighted phase functions and the codification of three multiplexed Fresnel lenses to discover their performance characteristics. This method is suitable for any complex multiplexed phase-diffractive element except for $M = 0$; however, $M = 0$ can be calculated by use of the polynomial fitting reported in this paper.

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