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Dynamic Analysis for Bose–Einstein Condensation in Quantum Cavity*

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Abstract For the two-level atoms system interacting with single-mode active field in a quantum cavity, the dynamics of the Bose–Einstein Condensation (BEC) is analyzed using an ordinary method suggested by authors to solve the system of Schrödinger representation in the Heisenberg representation. The wave function of the atoms is given. The stability factor determining the BEC and the selection rules of the quantum transition are solved.

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Key words: Bose–Einstein condensation, quantum cavity, representation, laser cooling

1 Introduction

Since the Bose–Einstein Condensation (BEC) of alkali metallic gas with weak interaction cooled by laser was experimentally realized in 1995,[1–3] the experimental researches have developed rapidly involving BEC, such as atom laser,[4] test of atomic matter-wave interference,[5] Brag reflection of atomic wave,[6] and mixture of four atomic waves.[7] Thus the dynamic analysis of BEC appeared to be very important.[8,9]

The Bose–Einstein Condensation in the quantum cavity is discussed in this paper using the ordinary method suggested by the authors for solving the wave function of the Schrödinger representation in the Heisenberg representation (see the appendix). Taking the interacting system composed of two-level Bose atoms and the single-mode active field as an example, the fully dynamics of BEC has been analyzed in detail. And the wave function of the Boson BEC is given. The stability factor determining the BEC and the selection rules of the quantum transition are solved.

2 Modeling

The system considered is the two-level Bose atoms interacting with the strong laser field in the single-mode quantized annular cavity. The one-particle Hamiltonian can be written as

\[ H = \frac{p^2}{2m} + \hbar \omega_a |e\rangle \langle e| + \hbar \omega a^\dagger a + \hbar \xi (|e\rangle \langle g| e^{i k_x} e^{-i\Omega t} + |g\rangle \langle e| e^{i k_x} + \hbar \xi (|e\rangle \langle g| e^{i k_x} e^{-i\Omega t} + |g\rangle \langle e| e^{i k_x} e^{i\Omega t}), \]

where \(|e\rangle\) and \(|g\rangle\) represent the excited state and the ground state of a two-level atom, \(\hbar \omega_a\) level difference, \(a^\dagger\) and \(a\) the creation and annihilation operators of the single-mode cavity field at the frequency of \(\omega\), and \(\xi\) the electric coupling coefficient between the cavity field and the atom system. The last item of Eq. (1) represents the interaction between the atoms and the strong laser field at frequency \(\Omega\). The Hamiltonian after second quantization is

\[ H' = \sum_p \hbar (\varepsilon_p + \omega_a) b_p^\dagger b_p + \sum_p \hbar \varepsilon_p b_p^\dagger b_g + \hbar \omega a^\dagger a + \hbar \xi \left( b_p^\dagger b_{g-p} + b_g^\dagger b_{p+} a^\dagger \right) \]

\[ + \hbar \xi \sum_p \left( b_p^\dagger b_{g-p} e^{-i\Omega t} + b_g^\dagger b_{p+} e^{i\Omega t}\right) , \]

where \(b_p, b_g, \) and \(b_p^\dagger, b_g^\dagger\) represent the annihilation and creation operators of the compound states of center of mass \(|p\rangle\) \((\equiv |e\rangle \otimes |p\rangle)\) and \(|g\rangle\) respectively.

The atom system condenses uniformly in the free space when BEC occurs. The creation and annihilation operators of the state \(|g, 0\rangle\) can be regarded as a constant under Bogliubov approximation, i.e.

\[ b_{g0}, b_{g0}^\dagger \rightarrow \sqrt{N_c} , \]

\(N_c\) is the number of condensed atoms.

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When $N_c$ is large enough, the items with $\sqrt{N_c}$ are retained, and the free field is neglected. Taking $b_{e\kappa} = b$, $b_{e\kappa}^\dagger = b^\dagger$, the effective Hamiltonian can then be approximately expressed as

$$H_{\text{eff}} = \hbar \omega a + h(\omega_a + \varepsilon_k) b^\dagger b + h \xi \sqrt{N_c} (b^\dagger a + a^\dagger b) + h \xi \sqrt{N_c} (b^\dagger e^{-i\Omega t} + be^{i\Omega t}),$$

(4)

where $\varepsilon_k = \hbar k^2/2m$

3 BEC of Atoms in Quantum Cavity

Now we analyze the system determined by the effective Hamiltonian (4) in the Heisenberg representation. The Heisenberg equations for $a(t)$ and $b(t)$ can be given as follows:

$$\dot{a}(t) = \frac{1}{\hbar} [a(t), H] = -i \omega_a(t) - i \xi \sqrt{N_c} b(t),$$

(5)

$$\dot{b}(t) = \frac{1}{\hbar} [b(t), H] = -i(\omega_a + \varepsilon_k) b(t) - i \sqrt{N_c} a(t) - i e^{-i\Omega t} N_c \xi.$$  

(6)

Let

$$B(t) = b(t) e^{i(\omega_a + \varepsilon_k)t},$$

(7)

and transform Eq. (5) into the integral form

$$a(t) = e^{-i\omega a(0)} - i \xi \sqrt{N_c} t \int_0^t B(t') e^{-i(\omega_a + \varepsilon_k)t'} e^{i\omega(t'-t)} dt'.$$

(8)

The following can then be obtained by substituting Eq. (8) into Eq. (6) and using Eq. (7)

$$\dot{B}(t) = -i \xi \sqrt{N_c} e^{-i(\omega - \omega_a - \varepsilon_k)t} a(0) - 2i N_c \int_0^t B(t') e^{i(\omega - \omega_a - \varepsilon_k)(t'-t)} dt',$$

(9)

whose Laplacian transformation is

$$\ddot{B}(p) = \int_0^\infty e^{-pt} B(t) dt = \frac{1}{p + \xi^2 N_c/p + i(\omega - \omega_a - \varepsilon_k)} \left[ b(0) - \frac{i \xi \sqrt{N_c} a(0)}{p + i(\omega - \omega_a - \varepsilon_k)} - \frac{i \xi \sqrt{N_c}}{p + i(\Omega - \omega_a - \varepsilon_k)} \right].$$

(10)

And the inverse transformation is obtained

$$b(t) = A_2(t) a(0) + B_2(t) b(0) + F_2(t),$$

(11)

where

$$A_2(t) = \frac{-\xi \sqrt{N_c}}{\sqrt{\Delta \omega^2 + 4\xi^2 N_c}} e^{(1)} + \frac{\xi \sqrt{N_c}}{\sqrt{\Delta \omega^2 + 4\xi^2 N_c}} e^{(2)},$$

(12)

$$B_2(t) = \frac{\Delta \omega + \sqrt{\Delta \omega^2 + 4\xi^2 N_c}}{2 \sqrt{\Delta \omega^2 + 4\xi^2 N_c}} e^{(1)} + \frac{-\Delta \omega + \sqrt{\Delta \omega^2 + 4\xi^2 N_c}}{2 \sqrt{\Delta \omega^2 + 4\xi^2 N_c}} e^{(2)},$$

(13)

$$F_2(t) = \frac{-\xi \sqrt{N_c} \left[ \Delta \omega + \sqrt{\Delta \omega^2 + 4\xi^2 N_c} \right]}{\sqrt{\Delta \omega^2 + 4\xi^2 N_c}} e^{(1)} + \frac{\xi \sqrt{N_c} \left[ \Delta \omega + \sqrt{\Delta \omega^2 + 4\xi^2 N_c} \right]}{\sqrt{\Delta \omega^2 + 4\xi^2 N_c}} e^{(2)} + \frac{4 \xi \sqrt{N_c} (\Omega - \omega)}{[\omega + \omega_a + \varepsilon_k - 2\Omega - \sqrt{\Delta \omega^2 + 4\xi^2 N_c}] [\omega + \omega_a + \varepsilon_k - 2\Omega + \sqrt{\Delta \omega^2 + 4\xi^2 N_c}]} e^{-i\Omega t}.$$  

(14)

Herein

$$e(1) = \exp \left\{ \frac{i}{2} \left[ (\omega + \omega_a + \varepsilon_k) + \sqrt{\Delta \omega^2 + 4\xi^2 N_c} \right] t \right\},$$

$$e(2) = \exp \left\{ \frac{-i}{2} \left[ (\omega + \omega_a + \varepsilon_k) + \sqrt{\Delta \omega^2 + 4\xi^2 N_c} \right] t \right\}, \quad \Delta \omega = \omega - \omega_a - \varepsilon_k.$$  

(15)

$a(t)$ can be solved by substituting Eqs. (11) into Eq. (6),

$$a(t) = A_1(t) a(0) + B_1(t) b(0) + F_1(t),$$

(16)

where

$$A_1(t) = \frac{-\Delta \omega + \sqrt{\Delta \omega^2 + 4\xi^2 N_c}}{2 \sqrt{\Delta \omega^2 + 4\xi^2 N_c}} e^{(1)} + \frac{\Delta \omega + \sqrt{\Delta \omega^2 + 4\xi^2 N_c}}{2 \sqrt{\Delta \omega^2 + 4\xi^2 N_c}} e^{(2)},$$

(17)
the wavefunction of the system at any time is generally expressed as

\[ B_1(t) = \frac{-c\sqrt{N_c}}{\sqrt{\Delta \omega^2 + 4\xi^2 N_c}} e(1) + \frac{c\sqrt{N_c}}{\sqrt{\Delta \omega^2 + 4\xi^2 N_c}} e(2) \]

\[ F_1(t) = \frac{2\xi N_c}{\sqrt{\Delta \omega^2 + 4\xi^2 N_c}} [2\Omega - \omega - \omega_a - \xi_k + \sqrt{\Delta \omega^2 + 4\xi^2 N_c}] e(1) \]

\[ + \frac{2\xi N_c}{\sqrt{\Delta \omega^2 + 4\xi^2 N_c}} [2\Omega - \omega - \omega_a - \xi_k - \sqrt{\Delta \omega^2 + 4\xi^2 N_c}] e(2) - \frac{\xi}{\sqrt{N_c}} e^{-i\Omega t} \]

\[ + \frac{\xi}{\sqrt{N_c}} \left[ \omega + \omega_a + \xi_k - 2\Omega - \sqrt{\Delta \omega^2 + 4\xi^2 N_c} \right] \left[ \omega + \omega_a + \xi_k - 2\Omega + \sqrt{\Delta \omega^2 + 4\xi^2 N_c} \right] e^{-i\Omega t}. \]  

4 Calculation of the System Wavefunction

It is now known that in Heisenberg representation

\[ a(t) = A_1(t)a(0) + A_2(t)b(0) + F_1(t), \quad b(t) = B_1(t)a(0) + B_2(t)b(0) + F_2(t). \]  

Wavefunction \( \psi(t) \) of Schrödinger representation can be obtained using the method detailed in the appendix. Assuming the initial state of the system is the total BEC state

\[ |\psi(0)\rangle = |0, N_c\rangle, \]

then

\[ |\psi(t)\rangle = \sum_{m,n} \langle m, n, t |\psi(0)\rangle |m, n\rangle. \]

Since

\[ |m, n, t\rangle \equiv |\alpha(t)\rangle \otimes |\beta(t)\rangle = |\alpha(t), \beta(t)\rangle \]

is the eigenfunction of \( a(t) \) and \( b(t) \) and so satisfies the following:

\[ a(0)|\alpha(t)\rangle = \alpha(t)|\alpha(t)\rangle, \quad b(0)|\beta(t)\rangle = \beta(t)|\beta(t)\rangle, \]

so

\[ a(t)|\alpha(t), \beta(t)\rangle = [A_1(t)a(0) + A_2(t)b(0) + F_1(t)]|\alpha(t), \beta(t)\rangle \]

\[ = [A_1(t)|\alpha(t)\rangle + A_1(t)|\beta(t)\rangle + F_1(t)]|\alpha(t), \beta(t)\rangle = m|\alpha(t), \beta(t)\rangle, \]

\[ b(t)|\alpha(t), \beta(t)\rangle = [A_2(t)a(0) + A_2(t)b(0) + F_2(t)]|\alpha(t), \beta(t)\rangle \]

\[ = [A_2(t)|\alpha(t)\rangle + B_2(t)|\beta(t)\rangle + F_2(t)]|\alpha(t), \beta(t)\rangle = n|\alpha(t), \beta(t)\rangle. \]

The following can be obtained,

\[ |m, n, t\rangle \equiv |\alpha(t), \beta(t)\rangle \equiv \frac{|m - F_1(t)|B_2(t) - |n - F_2(t)|B_1(t)}{B_2(t)A_1(t) - B_1(t)A_2(t)} \otimes \frac{|m - F_1(t)|A_2(t) - |n - F_2(t)|A_1(t)}{A_2(t)B_1(t) - A_1(t)B_2(t)}. \]

Substituting Eq. (27) into Eq. (22), the wavefunction of the system is given

\[ |\psi(t)\rangle = \sum_{m,n} \langle m, n, t |0, N_c\rangle |m, n\rangle = |F_1(t) + N_cB_1(t), F_2(t) + N_cB_2(t)\rangle. \]

5 Discussion of the Transition Probability

If we let the eigenfunction of \( H \) in Schrödinger representation be

\[ |m, n\rangle = |m\rangle_a \otimes |n\rangle_f = \frac{1}{\sqrt{m!}} b^m |0\rangle_a \otimes \frac{1}{\sqrt{n!}} a^\dagger |0\rangle_f, \]

and assume the initial state of the system is total BEC, the number of Bose atoms is \( N \), then there will be

\[ |\psi(0)\rangle = |0, N\rangle, \]

the wavefunction of the system at any time is generally expressed as

\[ |\psi(t)\rangle = U(t)|\psi(0)\rangle = U(t)|0, N\rangle. \]

Transition probability \( |\langle m, n |\psi(t)\rangle|^2 \) of state \( |\psi(t)\rangle \) at any time can be obtained by using \( a \) and \( b \) in Heisenberg representation.
It is known that

\[ U(t)|0, 0\rangle = e^{iHt}|0, 0\rangle , \tag{32} \]

so one has

\[ \langle m, n|\psi(t)\rangle = \langle 0, 0|U^\dagger(t)\frac{a_m(0)b^n(0)}{\sqrt{m!n!}} U(t)|0, N\rangle = \langle 0, 0|\frac{a_m(t)b^n(t)}{\sqrt{m!n!}} |0, N\rangle \]

\[ = \frac{1}{\sqrt{m!n!}} \langle 0, 0| \left[ A_1(t)a(0) + B_1(t)b(0) + F_1(t)\right]^m[A_2(t)a(0) + B_2(t)b(0) + F_2(t)]^n|0, N\rangle \]

\[ = \sqrt{m!n!} \langle 0, 0| \sum_k \sum_l \sum_q \sum_r \frac{q!r!(m-k)!(n-l)!(k-q)!(l-r)!}{k!(N-k)!} \times A_1^r(t)A_2^m(t)a^{q+r}(0)B_1^{l-r}(t)B_2^{k-q-r}(0)F_1^{m-k}(t)F_2^{n-l}(t)|0, N\rangle . \tag{33} \]

The selection rule is therefore

\[ q + r = N , \quad l + k - q - r = 0 , \tag{34} \]

and \( q \leq k, r \leq l. \)

The selection rule can then be written as

\[ k = q , \quad r = l , \quad l + k = N . \tag{35} \]

The probability of the state \(|m, n\rangle\) at any time can be obtained,

\[ P \equiv |\langle m, n|\psi(t)\rangle|^2 = \sqrt{m!n!}\sqrt{N!} \times \sum_k \sum_l \frac{F_1^{m-k}F_2^{n-k-N}A_1^l(t)A_2^{l-k}(t)}{k!(N-k)!} . \tag{36} \]

If the initial state of the system is totally the BEC state, the transition probability of a quantum state at any time reflects the stability of atoms in BEC state.

**Appendix**

It is found easier to study the behavior of (BEC) atoms in cavity in Heisenberg representation. In fact, it is very effective to solve problems in Heisenberg representation for many quantum systems. However, the problem is that it is not very convenient to investigate such problems as the transition probability of similar particles in Heisenberg representation. It is obviously significant to establish an ordinary method for solving the wave function of Schrödinger representation in Heisenberg representation.

Suppose the Hamiltonian of a quantum system under investigation is \( H = H(a, b) , \) and \( a, b \) are the annihilating operators of two types of particles in the system, the Heisenberg equation is then satisfied in the Heisenberg representation

\[ \dot{a}(t) = \frac{1}{\hbar}[a(t), H] , \]

\[ a(t)|\lambda_1, \lambda_2, t\rangle = U^\dagger a(0)U(t)U^\dagger|\lambda_1, \lambda_2, t\rangle = |\lambda_1, \lambda_2, \psi(t)\rangle , \]

\[ b(t)|\lambda_1, \lambda_2, t\rangle = U^\dagger b(0)U(t)U^\dagger|\lambda_1, \lambda_2\rangle = |\lambda_2, \lambda_1, \psi(t)\rangle . \tag{A9} \]

We can also prove that if there is no degeneracy, and \(|\lambda_1, \lambda_2, t\rangle\) is also the common eigenfunction of \( a(t), b(t) \), then

\[ |\lambda_1, \lambda_2, t\rangle = U^\dagger(t)|\lambda_1, \lambda_2\rangle . \tag{A10} \]
Summing up the above, the wave function of Schrödinger representation can be written as
\[ |\psi(t)\rangle = \int \langle \lambda_1, \lambda_2, t | \psi(0) | \lambda_1, \lambda_2 \rangle \, d\lambda_1 \, d\lambda_2, \] (A11)
and the way of determining \(|\lambda_1, \lambda_2, t\rangle\) is as follows.

For
\[ a(0) |\alpha(t)\rangle = |\alpha(t)\rangle |\alpha(t)\rangle, \quad b(0) |\beta(t)\rangle = |\beta(t)\rangle |\beta(t)\rangle, \] (A12)
and \(|\alpha(t) \otimes \beta(t)\rangle = |\alpha(t)\rangle |\beta(t)\rangle\) is the common eigenfunction of \(a(t)\) and \(b(t)\), then
\[ a(t) |\alpha(t), \beta(t)\rangle = [A_1(t) a(0) + B_1(t) b(0) + C_1(t)] |\alpha(t), \beta(t)\rangle \]
\[ = |\lambda_1 |\alpha(t), \beta(t)\rangle, \] (A13)
and similarly,
\[ b(t) |\alpha(t), \beta(t)\rangle = [A_2(t) a(0) + B_2(t) b(0) + C_2(t)] |\alpha(t), \beta(t)\rangle \]
\[ = |\lambda_2 |\alpha(t), \beta(t)\rangle, \] (A14)
\[ a(t) = \frac{\lambda_1 - C_2(t) B_1(t) - (\lambda_1 - C_1(t)) B_2(t)}{B_1(t) A_2(t) - A_1(t) B_2(t)}, \quad \beta(t) = \frac{\lambda_2 - C_2(t) A_1(t) - (\lambda_1 - C_1(t)) A_2(t)}{B_1(t) A_2(t) - A_1(t) B_2(t)}. \] (A15)
The wave function of Schrödinger representation can be obtained by substituting \(|\lambda_1, \lambda_2, t\rangle = |a(t) \otimes \beta(t)\rangle\) into Eq. (A11).

References