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Squeezing in the Real and Imaginary Spin Coherent States *

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We study spin squeezing properties in the real and imaginary spin coherent states. We obtain analytical expressions of two spin squeezing parameters via a novel ladder operator formalism of the spin coherent state and the generation function method. Numerical results of the spin squeezing properties are discussed in detail, and the real and imaginary spin coherent state can be spin squeezed over a large range of parameters.

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There has been great interest in the study of superpositions of coherent states in the quantized electromagnetic field[1] and quantized motion of the centre of mass of a trapped ion.[2] These states exhibit nonclassical properties such as oscillations of photon number distribution, antibunching effects and quadrature squeezing. Two types of superposition states, superpositions of the bosonic coherent states $|\alpha\rangle$ and $|-\alpha\rangle$[1] or superpositions of $|\alpha\rangle$ and $|\alpha^*\rangle$,[3] have been considered.

Squeezing of light (boson squeezing) has been well studied,[4−7] and squeezing in spin systems has recently received much attention.[8−33] Spin squeezed states are quantum correlated states with reduced fluctuations in one of the collective spin components, with possible applications in atomic interferometers and high precision atomic clocks. Interestingly, it was found that spin squeezing is closely related to and implies quantum entanglement,[30−33] a key ingredient in quantum information theory. The close relation between spin squeezing and entanglement enhances the importance of spin squeezing.

Now we replace the bosonic coherent state $|\alpha\rangle$ by the spin coherent state (SCS)[34] $|\eta\rangle(\eta = |\eta|e^{i\gamma})$ in the superpositions of coherent states $|\alpha\rangle \pm |\alpha^*\rangle$, and obtain the superpositions of two SCSs,

$$|\eta\rangle_\pm = N_\pm(|\eta\rangle \pm |\eta^*\rangle), \tag{1}$$

where $N_\pm$ is a normalization factor given by

$$N_\pm = \left[2 \pm (\langle\eta|\eta^*\rangle + \langle\eta^*|\eta\rangle)\right]^{-1/2}, \tag{2}$$

with $\langle\eta^*|\eta\rangle = (1+\eta^2)^{2j}/(1+|\eta|^2)^{2j}$. The states $|\alpha\rangle \pm |\alpha^*\rangle$ are named the real and imaginary coherent states,[3] so we may call the above states $|\eta\rangle_\pm$ the real and imaginary spin coherent states. Agarwal *et al.*,[35] Gerry and Grobe[36] and Recamier *et al.*[37] have introduced and studied the generation schemes and some properties of the superpositions of SCSs. Here we study their spin squeezing properties.[8] In addition, several real and imaginary quantum states have been studied in the literature.[3,38]

There are several definitions of spin squeezing,[8,9,29,30] and which one is the best is still an unsolved issue. Typically, there are two types of spin squeezing quantified by the following two spin squeezing parameters:

$$\begin{aligned}\xi_1^2 &= \frac{2(\Delta S_{\boldsymbol{n}_\perp})^2}{J} = \frac{4(\Delta S_{\boldsymbol{n}_\perp})^2}{N},\\ \xi_2^2 &= \frac{N(\Delta S_{\boldsymbol{n}_\perp})^2}{|\langle \boldsymbol{S}\cdot\boldsymbol{n}\rangle|^2}.\end{aligned} \tag{3}$$

Here the subscript $\boldsymbol{n}_\perp$ refers to an axis perpendicular to the mean spin direction $\boldsymbol{n} = \langle\boldsymbol{S}\rangle/\sqrt{\langle\boldsymbol{S}\rangle\cdot\langle\boldsymbol{S}\rangle}$, where the minimal value of the variance $(\Delta S)^2$ is obtained, $J = N/2$, and $S_{\boldsymbol{n}_\perp} = \boldsymbol{S}\cdot\boldsymbol{n}_\perp$. The inequality $\xi_1^2 < 1$ ($\xi_2^2 < 1$) indicates that the system is spin squeezed. The first parameter is from Kitagawa and Ueda,[8] and the second one from Wineland *et al.*[9]

Let us first introduce the SCS. We work in the $(2j+1)$-dimensional angular momentum Hilbert space $\{|j,m\rangle;\ m = -j,\ldots,+j\}$. It is convenient to define the *number* operator $\mathcal{N} = J_z + j$ and the *number states* as

$$\begin{aligned}&|n\rangle \equiv |j, -j+n\rangle,\\ &\mathcal{N}|n\rangle = n|n\rangle.\end{aligned} \tag{4}$$

The SCS is defined in this Hilbert space and given by[34]

$$|\eta\rangle = (1+|\eta|^2)^{-j}\sum_{n=0}^{2j}\binom{2j}{n}^{1/2}\eta^n|n\rangle, \tag{5}$$

where the parameter η is complex. Formally, the SCS is exactly of the form of the binomial state.[39] From

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the definition, we can obtain the two ladder operator formalisms of the SCS,

$$S_-|\eta\rangle = \eta(2j - \mathcal{N})|\eta\rangle, \tag{6}$$

$$S_+|\eta\rangle = \eta^{-1}\mathcal{N}|\eta\rangle, \tag{7}$$

where $S_\pm = S_x \pm iS_y$. The second one derived by us is, to our knowledge, completely new, and is very useful for obtaining analytical results of the spin squeezing.

According to the definitions of spin squeezing, we first need to know the mean spin direction determined by the expectation values $\langle S_\alpha\rangle$ for $\alpha \in \{x, y, z\}$. The mean spin direction $\boldsymbol{n}_1$ and the other two directions $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ perpendicular to $\boldsymbol{n}_1$ can be written in spherical coordinates as

$$\begin{pmatrix}\boldsymbol{n}_1\\ \boldsymbol{n}_2\\ \boldsymbol{n}_3\end{pmatrix} = \begin{pmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ -\sin\phi & \cos\phi & 0\\ -\cos\theta\cos\phi & -\cos\theta\sin\phi & \sin\theta\end{pmatrix}, \tag{8}$$

where θ and ϕ are the polar and azimuthal angles, respectively, and are given by $\theta = \arccos(\langle S_z\rangle/\sqrt{\langle S_x\rangle^2 + \langle S_y\rangle^2 + \langle S_z\rangle^2})$, $\phi = \arctan(\langle S_y\rangle/\langle S_x\rangle)$. The expressions for $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ are also given above.

Secondly, we need to compute the following variance values:[33]

$$(\Delta S_{\boldsymbol{n}_\perp})^2 = \langle S^2_{\boldsymbol{n}_\perp}\rangle = \langle S^2_{\boldsymbol{n}_{\perp 2}} + S^2_{\boldsymbol{n}_{\perp 3}}\rangle - \sqrt{(\langle S^2_{\boldsymbol{n}_{\perp 2}} - S^2_{\boldsymbol{n}_{\perp 3}}\rangle)^2 + \langle [S_{\boldsymbol{n}_{\perp 2}}, S_{\boldsymbol{n}_{\perp 3}}]_+\rangle^2} \tag{9}$$

with

$$\begin{aligned} S_{\boldsymbol{n}_{\perp 2}} &= -S_x\sin\phi + S_y\cos\phi,\\ S_{\boldsymbol{n}_{\perp 3}} &= -S_x\cos\theta\cos\phi - S_y\cos\theta\sin\phi\\ &\quad + S_z\sin\theta. \end{aligned} \tag{10}$$

To determine the spin squeezing we need to know the expectation values of the operators S_α, $S_\alpha^2, S_xS_y + S_yS_x$, $S_yS_z + S_zS_y$, and $S_zS_x + S_xS_z$. These can be written in terms of the operators $S_\pm$ and the number operator $\mathcal{N}$ as

$$\begin{aligned} S_x &= \frac{S_+ + S_-}{2}, \quad S_y = \frac{S_+ - S_-}{2i}, \quad S_z = \mathcal{N} - j,\\ S_x^2 &= \frac{1}{4}[2j(2\mathcal{N}+1) - 2\mathcal{N}^2 + S_+^2 + S_-^2],\\ S_y^2 &= \frac{1}{4}[2j(2\mathcal{N}+1) - 2\mathcal{N}^2 - S_+^2 - S_-^2],\\ S_z^2 &= \mathcal{N}^2 - 2j\mathcal{N} + j^2,\\ S_xS_y + S_yS_x &= \frac{S_+^2 - S_-^2}{2i},\\ S_yS_z + S_zS_y &= \frac{(2\mathcal{N}-1)S_+ - S_-(2\mathcal{N}-1)}{2i}\\ &\quad - \frac{j(S_+ - S_-)}{i},\\ S_zS_x + S_xS_z &= \frac{(2\mathcal{N}-1)S_+ + S_-(2\mathcal{N}-1)}{2}\\ &\quad - j(S_+ + S_-). \end{aligned} \tag{11}$$

From the above equations, we need to find the expectation values $\langle\mathcal{N}^k\rangle (k = 1, 2)$, which can be conveniently obtained by the generation function method. The generation function of the real and imaginary SCS is given by

$$\begin{aligned} G(\lambda) &= \langle\eta_\pm|\lambda^{\mathcal{N}}|\eta_\pm\rangle = {N_\pm}^{-2}[2g(\lambda) \pm \tilde{g}(\lambda) \pm \tilde{g}^*(\lambda)],\\ g(\lambda) &= \langle\eta|\lambda^{\mathcal{N}}|\eta\rangle = (1 + |\eta|^2)^{-2j}(1 + \lambda|\eta|^2)^{2j},\\ \tilde{g}(\lambda) &= \langle\eta|\lambda^{\mathcal{N}}|\eta\rangle = (1 + |\eta|^2)^{-2j}(1 + \lambda\eta^2)^{2j}. \end{aligned} \tag{12}$$

The factorial moments $F_k = d^kG(\lambda)/d\lambda^k|_{\lambda=1}$ then follow from the generation function

$$\begin{aligned} F_k &= {N_\pm}^{-2}[2f_k \pm \tilde{f}_k \pm \tilde{f}_k^*],\\ f_k &= \frac{|\eta|^{2k}(2j)!}{(1 + |\eta|^2)^k(2j - k)!},\\ \tilde{f}_k &= \frac{\eta^{2k}(1 + \eta^2)^{2j-k}(2j)!}{(1 + |\eta|^2)^{2j}(2j - k)!}. \end{aligned} \tag{13}$$

The expectation values $\langle\mathcal{N}^k\rangle$ $(k = 1, 2)$ can be expressed in terms of factorial moments as

$$\langle\mathcal{N}\rangle = F_1, \quad \langle\mathcal{N}^2\rangle = F_2 + F_1. \tag{14}$$

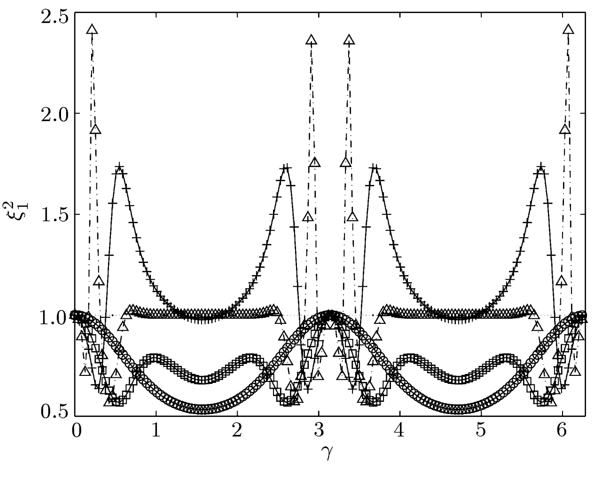

Fig. 1. Spin squeezing parameter ξ_1^2 of the real SCS versus γ for different values of $|\eta|$: $|\eta| = 0.1$ (circle line), $|\eta| = 0.2$ (square line), $|\eta| = 0.3$ (plus line), and $|\eta| = 0.5$ (triangle line).

From Eq. (11), we also need to know the following expectation values, $\langle S_+\rangle, \langle S_+^2\rangle$, and $\langle(2\mathcal{N} - 1)S_+\rangle$. From Eq. (7), we immediately have

$$\begin{aligned} S_+^2|\eta\rangle &= \eta^{-2}\mathcal{N}(\mathcal{N} - 1)|\eta\rangle,\\ (2\mathcal{N} - 1)S_+|\eta\rangle &= \eta^{-1}(2\mathcal{N} - 1)\mathcal{N}|\eta\rangle. \end{aligned} \tag{15}$$

The above equation and Eq. (7) show that the expectation values $\langle S_+\rangle, \langle S_+^2\rangle$, and $\langle(2\mathcal{N} - 1)S_+\rangle$ can also be obtained from the factorial moments. They are explicitly given by

$$\begin{aligned}\langle S_+\rangle &= \mathcal{N}_\pm^{-2}\Big[\Big(\frac{1}{\eta}+\frac{1}{\eta^*}\Big)f_1 \pm \frac{1}{\eta}\tilde{f}_1 \pm \frac{1}{\eta^*}\tilde{f}_1^*\Big],\\ \langle S_+^2\rangle &= \mathcal{N}_\pm{}^{-2}\Big[\Big(\frac{1}{\eta^2}+\frac{1}{\eta^{*2}}\Big)f_2 \pm \frac{1}{\eta^2}\tilde{f}_2\\ &\quad \pm \frac{1}{\eta^{*2}}\tilde{f}_2^*\Big],\\ \langle(2\mathcal{N}-1)S_+\rangle &= \mathcal{N}_\pm{}^{-2}\Big[\Big(\frac{1}{\eta}+\frac{1}{\eta^*}\Big)(2f_2+f_1)\\ &\quad \pm \frac{1}{\eta}(2\tilde{f}_2+\tilde{f}_1) \pm \frac{1}{\eta^*}(2\tilde{f}_2^*+\tilde{f}_1^*)\Big].\end{aligned} \tag{16}$$

After some substitutions, we can obtain expressions for the squeezing parameters of the real and imaginary SCS, and it is interesting to see that these

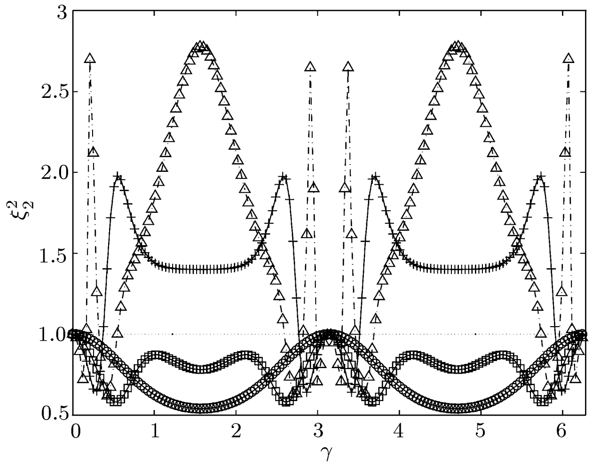

Fig. 2. Spin squeezing parameter ξ_2^2 of the real SCS versus γ for different values of $|\eta|$: $|\eta| = 0.1$ (circle line), $|\eta| = 0.2$ (square line), $|\eta| = 0.3$ (plus line), and $|\eta| = 0.5$ (triangle line).

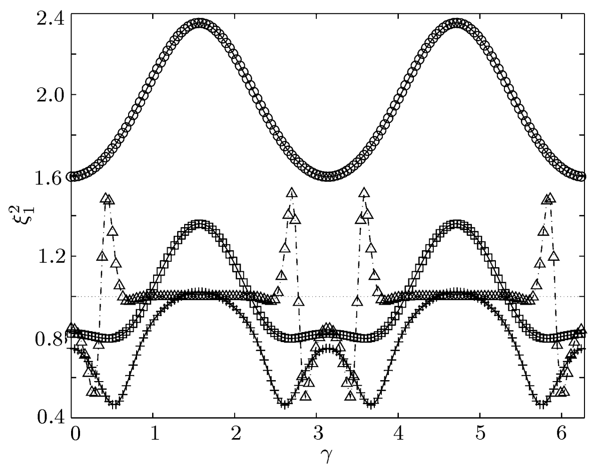

Fig. 3. Spin squeezing parameter ξ_1^2 of the imaginary SCS versus γ for different values of $|\eta|$: $|\eta| = 0.1$ (circle line), $|\eta| = 0.2$ (square line), $|\eta| = 0.3$ (plus line), and $|\eta| = 0.5$ (triangle line).

parameters can be solely determined by the first and second-order factorial moments. Having obtained the analytical expressions, we next perform the numerical calculations of the spin squeezing.

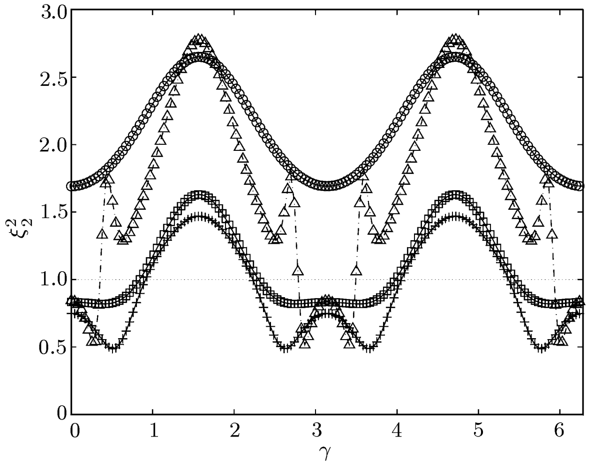

Fig. 4. Spin squeezing parameter ξ_2^2 of the imaginary SCS versus γ for different values of $|\eta|$: $|\eta| = 0.1$ (circle line), $|\eta| = 0.2$ (square line), $|\eta| = 0.3$ (plus line), and $|\eta| = 0.5$ (triangle line).

In Fig. 1 we plot the spin squeezing parameter ξ_1^2 of the real SCS versus γ for different values of $|\eta|$. For $\gamma = 0$, the real SCS exhibits no spin squeezing as it reduces to the SCS in this situation and $\xi_1^2 = 1$ for the SCS. For small values of $|\eta|$ (for instance $\eta = 0.1$), the real SCS is spin squeezed except at the points $\eta = n\pi$, $n \in \{0, 1, 2, \ldots\}$. For larger $|\eta|$, the range of spin squeezing is reduced. Thus, smaller values of $|\eta|$ are good for spin squeezing of the real SCS. The spin squeezing parameter ξ_2^2 displays the similar behaviour to that of ξ_1^2, as shown in Fig. 2. However, there also exist clear differences between them. For the case of $\eta = 0.3$, parameter ξ_1^2 is less than 1 over a small range of γ between $\gamma = 1$ and 2. However, parameter ξ_1^2 is larger than 1 in this range, indicating that the generation of spin squeezing according to parameter ξ_2^2 is more difficult than that according to parameter ξ_1^2.

We then consider the spin squeezing of the imaginary SCS, and the numerical results for the parameters ξ_1^2 and ξ_2^2 are shown in Figs. 3 and 4, respectively. We first note that, in the limit of $\gamma = 0$, there exists spin squeezing for larger $|\eta|$. We emphasize that we must take the limit to obtain a physical state, since the imaginary SCS with $\gamma = 0$ is not a physical state. For $\eta = 0.1$, the imaginary SCS is not spin squeezed; however, for $\eta = 0.3$, the state is spin squeezed over a large range of parameters of γ. We see that the spin squeezing properties of the imaginary SCS are distinct from those of the real SCS.

In conclusion, we have studied the spin squeezing properties of the real and imaginary spin coherent states. These types of superposition states can be generated by the state reduction techniques.[38] By using the two ladder operator formalisms of the SCS and the generation function method, we have readily obtained the analytical expressions for the spin squeezing parameters ξ_1^2 and ξ_2^2, which facilitates our discussions of spin squeezing. The method we have demonstrated to obtain the analytical results of spin squeezing is applicable to more general states, other than the real and imaginary SCSs. For example, we can apply the method to multi-component superpositions of SCSs such as $|\Psi\rangle = \sum_{n=1}^{N} c_n|\eta_n\rangle$. We have numerically computed the spin squeezing of the real and imaginary SCS and found that these states can be spin squeezed over a large range of parameters.

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