AIP | Journal of Mathematical Physics

Extremal asymmetric universal cloning machines

Mingming Jiang and Sixia Yu

Citation: J. Math. Phys. **51**, 052306 (2010); doi: 10.1063/1.3380527
View online: http://dx.doi.org/10.1063/1.3380527
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v51/i5
Published by the American Institute of Physics.

JOURNAL OF MATHEMATICAL PHYSICS **51**, 052306 (2010)

Extremal asymmetric universal cloning machines

Mingming Jiang[1,2,a)] and Sixia Yu[1]

[1]*Department of Modern Physics, University of Science and Technology of China, Hefei 230027, People's Republic of China*

[2]*Laboratory of Excited State Processes, Changchun Institute of Optics, Fine Mechanics, and Physics, Chinese Academy of Sciences, Changchun 130033, People's Republic of China*

(Received 7 May 2009; accepted 12 March 2010; published online 25 May 2010)

The trade-offs among various output fidelities of asymmetric universal cloning machines are investigated. First we find out all the attainable optimal output fidelities for the 1 to 3 asymmetric universal cloning machine and it turns out that there are two kinds of extremal machines which have to cooperate in order to achieve some of the optimal output fidelities. Second we construct a family of extremal cloning machines that includes the universal symmetric cloning machine as well as an asymmetric 1 to $1+N$ cloning machine for qudits with two different output fidelities such that the optimal trade-off between the measurement disturbance and state estimation is attained in the limit of infinite N. © *2010 American Institute of Physics.* [doi:10.1063/1.3380527]

A single quantum can neither be cloned[1] nor be broadcasted,[2] but it can be approximately cloned universally for qubits[3,4] and for qudits,[5–7] or probabilistically,[8] symmetrically or asymmetrically,[9–11] and experimentally.[12] The quantitative boundary between what is possible and impossible hinted by the no-cloning theorem has been explored in a few cases including the optimal symmetric cloning machines[6,9] and the optimal $1\mapsto 2$ and $1\mapsto 3$ asymmetric cloning machines.[13]

A universal $1\mapsto N$ cloning machine is a quantum mechanical process with one input and N outputs with the fidelity between each output state and the input state being independent of the input state. Symmetric cloning machines, which are special cases of the asymmetric cloning machines, are characterized by the unique maximal attainable output fidelity. For asymmetric cloning machines optimal trade-offs among the output fidelities in certain range of values have been explored.[13] In addition, a 1 to $1+N$ asymmetric cloning machine with two different output fidelities for qubits has also been constructed which, in the large N limit, balances the inequality of measurement disturbance and state estimation.[14]

In this letter we shall present at first the complete trade-off among three output fidelities of the universal 1 to 3 cloning machine for qudits. It turns out that there are two kinds of extremal cloning machines and for some range of output fidelities the two extremal cloning machines must cooperate to attain the optimal fidelities instead of a single "optimal" cloning machine. Second we construct also a 1 to $1+N$ cloning machine for qudits, which belongs to a family of extremal cloning machines in the symmetric subspace, that saturates Banaszek's inequality of measurement disturbance and state estimation.

In the following we consider only qudits, i.e., d-level systems whose Hilbert space is spanned by $\{|n\rangle\}_{n=0}^{d-1}$. Let us start with a trivial case to establish some notations, namely, a 1 to 1 universal cloning machine, which can be represented by a completely positive map $\psi\mapsto\mathcal{C}_1(\psi)$, where ψ represents the density matrix of a pure state $|\psi\rangle$ of a single qudit which is labeled by A. The output fidelity, taking into account of the universality, reads

[a)]Electronic mail: jmm5454@mail.ustc.edu.cn.

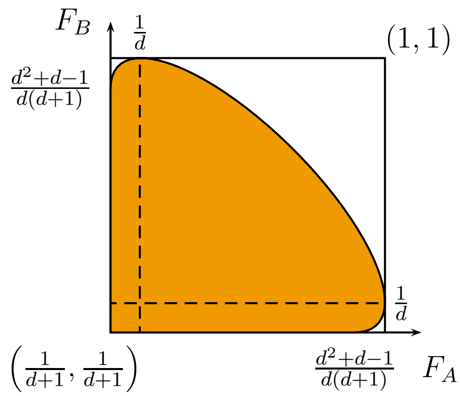

FIG. 1. (Color online) The trade-off between two output fidelities of 1 to 2 asymmetric cloning machine. The shaded area which is bounded by two axes and part of an ellipse contains all possible output fidelities.

$$F_A = \int \mathrm{Tr}(\psi \mathcal{C}_1(\psi))d\psi = \frac{d+f_A}{d(d+1)}, \tag{1}$$

where $f_A=\mathrm{Tr}(Q_{RA}\Phi_{RA})$ with $Q_{RA}=\mathcal{I}_R\otimes\mathcal{C}_1(\Phi_{RA})$ being a subnormalized state (Tr $Q_{RA}=d$) of the composite system of a reference qudit R and the original qudit A and Φ_{RA} denoting the density matrix of a (subnormalized) maximally entangled state $|\Phi\rangle=\Sigma_n|nn\rangle$ of the composite system RA. It is obvious that the output fidelity F_A ranges from $1/(d+1)$ to 1 because f_A takes values from 0 to d^2. The maximal output fidelity arises from the identity map $\mathcal{I}(\psi)=\psi$ and the minimal fidelity arises from the fact that the cloning machine must be a physical process allowed by the principle of quantum mechanics, i.e., $\mathcal{C}(\psi)$ is a completely positive map. In the case of $d=2$ the minimal output fidelity is achieved by the optimal universal NOT gate.

The situation is similar for cloning machines producing two or more copies. Let us consider now a $1\mapsto 2$ universal cloning machine, which can be represented by a completely positive map $\mathcal{C}_2$ from $\mathcal{H}_A$ to $\mathcal{H}_A\otimes\mathcal{H}_B$. Its two output fidelities F_A and F_B are determined by the expectation values f_A and f_B of two observables Φ_{RA} and Φ_{RB} in the subnormalized state $Q_{RAB}=\mathcal{I}_R\otimes\mathcal{C}_2(\Phi_{RA})$. Thus the bound of the optimal output fidelities is bounded by all the possible expectation values of two observables Φ_{RA} and Φ_{RB} when the state runs over all possible states of composite system RAB.

Obviously the range of two observables Φ_{RA} and Φ_{RB} is spanned by $2d$ states $|\Phi\rangle_{RA}|k\rangle_B$ and $|\Phi\rangle_{RB}|k\rangle_A$ with $k=0,1,\ldots,d-1$, from which an orthonormal basis can be constructed,

$$|\phi_k^{\pm}\rangle = \frac{1}{\sqrt{2(d\pm 1)}}(|\Phi\rangle_{RA}|k\rangle_B \pm |\Phi\rangle_{RB}|k\rangle_A). \tag{2}$$

It is not complete thus $\Sigma_k\phi_k^+ + \phi_k^- \le \mathbf{I}_3$, where $\phi_k^{\pm}$ denotes the projector of the corresponding state and $\mathbf{I}_3$ is the identity matrix for 3-qudit. When averaged in an arbitrary 3-qudit state Q_{RAB} with normalization Tr $Q_{RAB}=d$, the incompleteness condition leads to

$$\frac{(\sqrt{f_A}+\sqrt{f_B})^2}{2(d+1)} + \frac{(\sqrt{f_A}-\sqrt{f_B})^2}{2(d-1)} \le d. \tag{3}$$

This inequality can be regarded as an uncertainty relationship between observables Φ_{RA} and Φ_{RB}. The expectation values that saturate the inequality Eq. (3) for a 3-qudit state corresponding to the optimal 1 to 2 asymmetric cloning machine[13] without the restriction that the coefficients be non-negative. Thus the trade-off between two output fidelities F_A and F_B can be plotted as in Fig. 1. It should be pointed out that given one of the output fidelities in the interval between $1-1/d(d+1)$ and 1, the other output fidelity assumes a minimal value which is greater than the minimal possible fidelity $1/(d+1)$.

Let us now consider a 1 to 3 asymmetric universal cloning machine, which can be represented by a quantum operation $\mathcal{C}_3$ with 1 input and 3 outputs. In this case three output fidelities F_A, F_B, and F_C are determined through Eq. (1) by the expectation values f_A, f_B, and f_C of three observables Φ_{RA}, Φ_{RB}, and Φ_{RC} in a 4-qudit state $Q_{RABC}=\mathcal{I}_R\otimes\mathcal{C}_3(\Phi_{RA})$ which is subnormalized as $\mathrm{Tr}\, Q_{RABC}=d$. To explore all the possible output fidelities, we shall at first find out all the possible expectation values of those three observables in the same state and then we construct asymmetric cloning machines that attain those optimal values.

At first we notice that the Hilbert space of 4-qudit can be decomposed into three orthogonal subspaces,

$$\mathcal{H}_4 = V_+ \oplus V_- \oplus V_0, \tag{4}$$

where the subspace V_0 is the orthogonal complement of $V_+\oplus V_-$ with subspaces $V_\pm$ spanned, respectively, by bases,

$$|\phi_{kl}^{a\pm}\rangle = \frac{\mathbf{I}_4 + \omega^a\mathbf{Y} + \omega^{2a}\mathbf{Y}^2}{\sqrt{3(d \pm (3\delta_{a0}-1))}}|\Phi\rangle_{RA}|\{kl\}_\pm\rangle_{BC}, \tag{5}$$

where $\mathbf{I}_4$ is the identity operator for 4-qudit and $\mathbf{Y}$ denotes the cyclic permutation operator acting only on three qudits A,B,C with effects $\mathbf{Y}|m,n,k\rangle_{ABC}=|k,m,n\rangle_{ABC}$ for arbitrary m,n,k and leaving the qudit R unchanged, and $|\{kl\}_\pm\rangle=(|kl\rangle\pm|lk\rangle)/\sqrt{2}$ for $k>l$ and $|\{kk\}_+\rangle=|kk\rangle$.

Subspace $V_+\oplus V_-$ is the range of three observables Φ_{RA}, Φ_{RB}, and Φ_{RC} and therefore all the expectation values of these three observables are zero in V_0. Furthermore, we have $\langle\phi_{kl}^{a+}|\Phi_{R\alpha}|\phi_{mn}^{b-}\rangle=0$ ($\alpha=A,B,C$). As a result all the attainable expectation values of three observables $\Phi_{R\alpha}(\alpha=A,B,C)$ are those convex combinations of these attainable values in pure states in $V_\pm$ and 0, the value attained in V_0. In other words if we have found out two sets of all the attainable expectation values under the pure states in subspaces $V_\pm$ then the complete set of attainable values is the convex hull of these two sets and 0.

For an arbitrary pure (subnormalized) state $|\psi_\pm\rangle$ in $V_\pm$ with $\langle\psi|\psi\rangle=d$ we denote $f_{\alpha\pm}=\langle\psi_\pm|\Phi_{R\alpha}|\psi_\pm\rangle$ for $\alpha=A,B,C$ and $\mathbf{f}_{A\pm}$ as a $d(d\pm1)/2$-dimensional complex vector whose components are $\langle\psi_\pm|\Phi\rangle_{RA}|\{kl\}_\pm\rangle_{BC}$ with $k,l=0,1,\ldots,d-1$ and similarly for $\mathbf{f}_{\mathbf{B}\pm}$ and $\mathbf{f}_{\mathbf{C}\pm}$. Obviously $f_{\alpha\pm}=|\mathbf{f}_{\alpha\pm}|^{\mathbf{2}}$ for all $\alpha=A,B,C$. Since $V_\pm$ is only a subspace one has

$$\sum_{a=0}^{2}\sum_{k\geq l}^{d-1}|\phi_{kl\pm}^a\rangle\langle\phi_{kl\pm}^a| \leq \mathbf{I}_4, \tag{6}$$

which leads to

$$\sum_{\alpha=A,B,C} f_{\alpha\pm} \mp \frac{|\mathbf{f}_{A\pm}+\mathbf{f}_{B\pm}+\mathbf{f}_{C\pm}|^2}{d\pm2} \leq d(d\mp1) \tag{7}$$

when averaged in the state $|\psi_\pm\rangle$, respectively. Given the lengths of three complex vectors $\mathbf{f}_A$, $\mathbf{f}_B$, and $\mathbf{f}_C$, the total length $|\mathbf{f}_A+\mathbf{f}_B+\mathbf{f}_C|$ is bounded above by $|\mathbf{f}_A|+|\mathbf{f}_B|+|\mathbf{f}_C|$ and bounded from below by the maximum among 0, $|\mathbf{f}_A|-|\mathbf{f}_B|-|\mathbf{f}_C|$, $|\mathbf{f}_B|-|\mathbf{f}_A|-|\mathbf{f}_C|$, and $|\mathbf{f}_C|-|\mathbf{f}_B|-|\mathbf{f}_A|$. Thus it follows from Eq. (7) that

$$x^2+y^2+z^2-\frac{(x+y+z)^2}{d+2} \leq d(d-1) \tag{8}$$

in the symmetric subspace V_+, where we have denoted $x=\sqrt{f_A}$, $y=\sqrt{f_B}$, and $z=\sqrt{f_C}$ for convenience, and in the antisymmetric subspace V_- the expectation values satisfy either any one of the following inequalities:

$$x^2+y^2+z^2+\frac{(x+y-z)^2}{d-2} \leq d(d+1), \tag{9a}$$

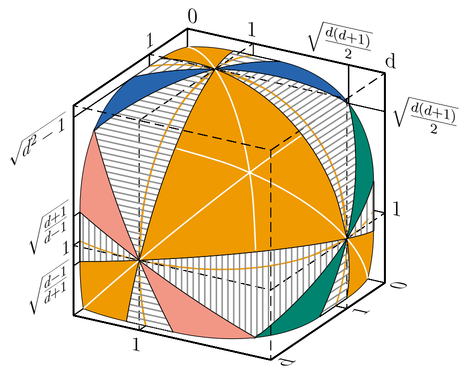

FIG. 2. (Color online) The convex hull of four ellipsoids with colored parts being the extremal points.

$$x^2+y^2+z^2+\frac{(x-y+z)^2}{d-2}\le d(d+1), \tag{9b}$$

$$x^2+y^2+z^2+\frac{(x-y-z)^2}{d-2}\le d(d+1), \tag{9c}$$

together with restrictions $z\ge x+y$, $y\ge x+z$, and $x\ge z+y$, respectively, or lie within the sphere,

$$x^2+y^2+z^2\le d(d+1) \tag{10}$$

restricted by the conditions

$$x\le y+z,\quad y\le x+z,\quad z\le x+y. \tag{11}$$

These bounds specify the range of all the possible expectation values of $\Phi_{R\alpha}$ $(\alpha=A,B,C)$ in pure states. Thus all the possible expectation values of three observables $\Phi_{R\alpha}$ $(\alpha=A,B,C)$ in arbitrary states are all possible convex combinations of those bounds, i.e., the boundary is the convex hull of those four ellipsoids defined in Eq. (8) and Eqs. (9a)–(9c) and the partial sphere in Eq. (10), which is explicitly plotted in the Fig. 2. We note that the restricted sphere equation (10) is contained in the convex hull for $d\ge 3$ and in the case of $d=2$ the boundary is the convex hull of Eqs. (8) and (10). Since the function $\sqrt{x}$ is a one-to-one concave function, the boundary for the fidelities F_α has essentially the same structure as the boundary for $\sqrt{f_\alpha}$ $(\alpha=A,B,C)$.

In the following we shall prove that the surface of the convex hull as plotted in Fig. 2 is attainable by explicitly constructing the universal cloning machines with the desired output fidelities. To do so we have only to construct the cloning machines that saturate those four inequalities Eqs. (8) and (9a)–(9c), respectively. We consider a system of five qudits labeled with A, B, C, E, and F and define two unitary evolutions as

$$U_\pm|m_A 0_{BCEF}\rangle=\sqrt{\frac{2}{d(d\pm 1)}}(\alpha+\beta\mathbf{Y}+\gamma\mathbf{Y}^2)$$

$$|m\rangle_A(|\Phi\rangle_{BE}|\Phi\rangle_{CF}\pm|\Phi\rangle_{CE}|\Phi\rangle_{BF}), \tag{12}$$

where $\mathbf{Y}$ is the permutation acting on ABC as before and α, β, and γ are real numbers satisfying

$$\alpha^2+\beta^2+\gamma^2\pm\frac{2}{d}(\alpha\beta+\beta\gamma+\gamma\alpha)=1. \tag{13}$$

It is easy to check that the cloning machines defined by $U_\pm$ are universal. For convenience we denote $x_\pm=d\alpha\pm(\beta+\gamma)$, $y_\pm=d\beta\pm(\alpha+\gamma)$, and $z_\pm=d\gamma\pm(\beta+\alpha)$.

We consider at first the cloning machine U_+. In the case of $x_+,y_+,z_+\geq 0$ we have $f_A=x_+^2$, $f_B=y_+^2$, and $f_C=z_+^2$ and the, inequality, Eq. (8) becomes an equality. Thus we have constructed an *extremal* cloning machines U_+ that saturates the inequality, Eq. (8). As will see in the following discussions the extremal cloning machines do not always produce the optimal output fidelities. In the case of non-negative α,β,γ the unitary evolution U_+ defines exactly the asymmetric cloning machine investigated in Ref. 13 with optimal output fidelities corresponding to the central golden area in Fig. 2. In the case of two negative and one positive coefficients among α,β,γ while keeping x_+,y_+,z_+ non-negative, U_+ also gives rise to the optimal cloning machines with fidelities corresponding to three small golden areas in Fig. 2. The boundaries of those four golden regions are $(d+1)x=y+z$, $(d+1)y=x+z$, and $(d+1)z=x+y$.

Next we consider the cloning machine U_-. Three output fidelities of the cloning machine U_- are $f_A=x_-^2$, $f_B=y_-^2$, and $f_C=z_-^2$, and they saturates the inequality Eq. (9a) in the case of x_-, $y_-\geq 0$, and $z_-\leq 0$. Similarly the inequalities Eqs. (9b) and (9c) are saturated by choosing x_-, $z_-\geq 0$ and $y_-\leq 0$ or y_-, $z_-\geq 0$ and $x_-\leq 0$. These machines therefore attain the optimal fidelities in the blue, green, and red regions in Fig. 2.

In the stripped white regions in Fig. 2 the optimal output fidelities are attained by neither of these two extremal machines $U_\pm$. Instead the optimal values can be achieved by a suitable cooperation of $U_\pm$. Since any value in the stripped white regions is a convex combination of the extremal values in the colored regions, it can be attained by mixing properly those extremal cloning machines achieving the extremal values. For example, let $(x,y,z)=p(x,y,z)_G+(1-p)\times(x,y',z)_B$ be an optimal value in a stripped white region, that is, a convex combination of two optimal values in the blue and golden regions. Let U_G and U_B be the extremal machines described above then by applying the machine U_G with probability q and U_B with probability $1-q$ we obtain the desired optimal fidelity (x,y,z) where q is uniquely determined by $(qy+(1-q)y')^2=py^2+(1-p)y'^2$.

At last we consider 1 to N asymmetric universal cloning machines which can be represented by a quantum operation $\mathcal{C}_N$ with one input and N outputs that are labeled from 1 to N. Each output fidelity F_n is determined through Eq. (1) by the expectation value f_n of the observable Φ_{0n} in the subnormalized state $Q_{0N}=\mathcal{I}_0\otimes\mathcal{C}_N(\Phi_{01})$. In what follows we shall find out a partial bound for the expectation values of Φ_{0k} and construct the cloning machine attaining this bound. A complete bound even in the simplest case $N=4$ is unattainable so far.

The range of N observables Φ_{0k} is spanned by the following Nd^{N-1} states (not normalized):

$$|\psi_\lambda^a\rangle=\mathbf{P}_a|\Phi\rangle_{01}|\lambda\rangle_{23\ldots N},\quad \mathbf{P}_a=\frac{1}{N}\sum_{k=0}^{N-1}\omega^{ka}\mathbf{X}^k, \tag{14}$$

where $\mathbf{X}$ is the permutation acting on the last N qudits according to $\mathbf{X}|n_1,n_2,\ldots,n_N\rangle=|n_2,n_3,\ldots,n_N,n_1\rangle$, $a=0,1,\ldots,N-1$, and $\{|\lambda\rangle_{23\ldots N}\}$ is an arbitrary basis for the last $N-1$ qudits. Let $|\psi\rangle$ be an arbitrary pure $(N+1)$-qudit state the Gramm matrix of these $Nd^{N-1}+1$ states $\{|\psi\rangle,|\psi_\lambda^a\rangle\}$ should be semipositive definite, i.e.,

$$\begin{pmatrix} d & \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3 & \cdots & \mathbf{f}_N \\ \mathbf{f}_1^\dagger & \mathrm{Tr}_1\,\mathbf{P_0} & 0 & 0 & \cdots & 0 \\ \mathbf{f}_2^\dagger & 0 & \mathrm{Tr}_1\,\mathbf{P}_1 & 0 & \cdots & 0 \\ \mathbf{f}_3^\dagger & 0 & 0 & \mathrm{Tr}_1\,\mathbf{P}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{f}_N^\dagger & 0 & 0 & 0 & \cdots & \mathrm{Tr}_1\,\mathbf{P}_{N-1} \end{pmatrix}\geq 0, \tag{15}$$

where $\mathbf{f}_{a+1}$ denotes a d^{N-1}-dimensional vector with components $\langle\psi|\psi_\lambda^a\rangle$ for $a=0,1,\ldots,N-1$.

By partitioning the Hilbert space of the last $N-1$ qudits into symmetric subspace, which is spanned by all the symmetric states $|\mathbf{n}\rangle_{23\ldots N}$, and its orthogonal complement, the Gramm matrix assumes a quasidiagonal form, and in the symmetric subspace the non-negativeness of the Gramm matrix gives rise to

$$\sum_{k=1}^{N} f_k - \frac{1}{d+N-1}\left(\sum_{k=1}^{N}\sqrt{f_k}\right)^2 \le d(d-1) \tag{16}$$

by noticing $N\,\mathrm{Tr}_1\,\mathbf{P}_0=d+N-1$ while $N\,\mathrm{Tr}_1\,\mathbf{P}_a=d-1$ in the symmetric subspace. Here we have denoted $f_k=|\mathbf{f}_k|^2$.

Let us now construct the cloning machine whose output fidelities saturate the inequality above. Consider the unitary evolution defined by

$$U_\alpha|m\rangle_1|0\rangle_{23\ldots N}|0\rangle_{2'3'\ldots N'} = \sum_{a=0}^{N-1}\frac{\alpha_a\mathbf{X}^a}{\sqrt{\binom{d+N-1}{N}}}|m\rangle_1\sum_{\mathbf{n}}|\mathbf{n}\rangle_{23\ldots N}|\mathbf{n}\rangle_{2'3'\ldots N'}, \tag{17}$$

with real numbers α_a satisfying

$$\sum_{a=0}^{N-1}\alpha_a^2 + \frac{2}{d}\sum_{a>b}^{N-1}\alpha_a\alpha_b = 1. \tag{18}$$

As long as $x_{a+1}=(d-1)\alpha_a+\Sigma_a\alpha_a\ge 0$ for all $a=0,1,\ldots,N-1$, the inequality Eq. (16) is saturated with fidelities given by $f_a=x_a^2$. Obviously the symmetric universal 1 to N cloning machine is a special case.

In addition if we take $\alpha_a=\beta/(d+N-1)$ for $a=1,2,\ldots,N-1$ and $\alpha_0=\alpha+\beta/(d+N-1)$ with α,β being non-negative, there are only two different output fidelities $f=(d\alpha+\beta)^2$ and $g=(\alpha+\beta)^2$. The normalization condition, Eq. (18), yields

$$(\sqrt{f}-\sqrt{g})^2 = (d-g)(d-1) + \frac{(d\sqrt{g}-\sqrt{f})^2}{d+N-1}, \tag{19}$$

which saturates the optimal trade-off between the information gain and state disturbance[14] when N tends to infinity. The last $N-1$ outputs with the same fidelity g provide the information gain because of the equivalency between the state estimation and symmetric cloning with an infinite outputs,[15] while the first output fidelity f characterizes the disturbance suffered in estimating the quantum state.

It should be pointed out that Eq. (16) need not to be satisfied by all the optimal output fidelities. That is, to say, there are some output fidelities that will fall outside the hype ellipsoid given by Eq. (16). Therefore, the cloning machine U_α does not always produce the optimal output fidelities. We believe that (without proof) when $\alpha_a\ge 0$ $(a=0,1,\ldots,N-1)$ the asymmetric cloning machine U_s is optimal which means the bound equation (16) holds true for this special range of output fidelities.

We acknowledge the financial support of NNSF of China (Grant No. 10675107).

[1] W. K. Wootters and W. H. Zurek, Nature (London) **299**, 802 (1982).
[2] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher, Phys. Rev. Lett. **76**, 2818 (1996).
[3] V. Bužek and M. Hillery, Phys. Rev. A **54**, 1844 (1996).
[4] N. Gisin and S. Massar, Phys. Rev. Lett. **79**, 2153 (1997).
[5] V. Bužek and M. Hillery, Phys. Rev. Lett. **81**, 5003 (1998).
[6] R. F. Werner, Phys. Rev. A **58**, 1827 (1998).
[7] H. Fan, K. Matsumoto, and M. Wadati, Phys. Rev. A **64**, 064301 (2001).
[8] L. M. Duan and G. C. Guo, Phys. Rev. Lett. **80**, 4999 (1998).
[9] C. S. Niu and R. B. Griffiths, Phys. Rev. A **58**, 4377 (1998).
[10] N. J. Cerf, Phys. Rev. Lett. **84**, 4497 (2000).
[11] N. J. Cerf, J. Mod. Opt. **47**, 187 (2000).

[12] A. Lamas-Linares, C. Simon, J. C. Howell, and D. Bouwmeester, Science **296**, 712 (2002); S. Fasel, N. Gisin, G. Ribordy, V. Scarani, and H. Zbinden, Phys. Rev. Lett. **89**, 107901 (2002); F. De Martini, D. Pelliccia, and F. Sciarrino, *ibid.* **92**, 067901 (2004); M. Ricci, F. Sciarrino, C. Sias, and F. DeMartini, *ibid.* **92**, 047901 (2004); W. T. M. Irvine, A. L. Linares, M. J. A. de Dood, and D. Bouwmeester, *ibid.* **92**, 047902 (2004).
[13] S. Iblisdir, A. Acin, N. J. Cerf, R. Filip, J. Fiuróšek, and N. Gisin, Phys. Rev. A **72**, 042328 (2005).
[14] K. Banaszek, Phys. Rev. Lett. **86**, 1366 (2001).
[15] D. Bruss, A. Ekert, and C. Macchiavello, Phys. Rev. Lett. **81**, 2598 (1998).